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Screening of mass singularities and finite soft-photon production rate in hot QCD

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A. Niégawa*

Department of Physics, Osaka City University

Sumiyoshi-ku, Osaka 558, Japan

Abstract

The production rate of a soft photon from a hot quark-gluon plasma is computed to leading order at logarithmic accuracy. The canonical hard-thermal-loop resummation scheme leads to logarithmically divergent production rate due to mass singularities. We show that these mass singularities are screened by employing the effective hard-quark propagator, which is obtained through resummation of one-loop self-energy part in a self-consistent manner. The damping-rate part of the effective hard-quark propagator, rather than the thermal-mass part, plays the dominant role of screening mass singularities. Diagrams including photon–(hard-)quark vertex corrections also yield leading contribution to the production rate.

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* E-mail: niegawa@sci.osaka-cu.ac.jp

I. INTRODUCTION

It has been established by Pisarski and Braaten and by Frenkel and Taylor [1,2] that, in perturbative thermal QCD, the resummations of the leading-order terms, called hard thermal loops, are necessary. In thermal massless QCD, we encounter the infrared and mass or collinear singularities. The hard-thermal-loop (HTL) resummed propagators soften or screen the infrared singularities, and render otherwise divergent physical quantities finite [3,4], if they are not sensitive to a further resummation of the corrections of $O(g^2T)$. There are some physical quantities which are sensitive to $O(g^2T)$ corrections, among those is the damping rate of a moving particle in a hot quark-gluon plasma. Much work has been devoted to this issue [1,5–7]. (For reviews of infrared and mass singularities in thermal field theory, we refer to [8,9].)

Among the thermal reactions, which are expected to serve as identifying the hot quark-gluon plasma is the soft-photon ($E = O(gT)$) production. This process is analyzed in [10,11] to leading order within the HTL resummation scheme. The conclusion is that the production rate is logarithmically divergent, owing to mass singularities.

The mass singularities found in [10,11] arise from bare (massless) hard-quark propagators that are on the mass-shell. This is a signal [12–14] of necessity of resummation for such propagators. Substituting the effective hard-quark propagators, $\hat{S}$s, which is obtained by resumming the one-loop self-energy part in a self-consistent manner (cf. e.g. [15]), for the bare propagators Ss, we show that the mass singularities are screened and the diverging factor in the production rate turns out to $\ln(g^{-1})$. This substitution violates the current-conservation condition. For recovering it, photon-quark vertex corrections should be taken into account. Among those is a set of diagrams that yields leading contribution to the production rate.

Here it is worth recording the relations between the differential rate $E\, dW/d^3p$ of a soft-photon [$P^\mu = (E, \mathbf{p})$, $E = O(gT)$] production, to be analyzed in this paper, and other quantities which are of interest in the literature. The traditionally defined production rate Γ_p is related to $E\, dW/d^3p$ as

$$\Gamma_p = (2\pi)^3 \frac{1}{E} \left(E \frac{dW}{d^3p} \right) . \tag{1.1}$$

The decay rate, Γ_d, of a soft photon in a hot quark-gluon plasma is related to Γ_p as

$$\Gamma_d = \frac{1}{2} \frac{1 + n_B(E)}{n_B(E)} \, \Gamma_p \simeq \frac{1}{2} \, \Gamma_p \, ,$$

where $n_B(E) = 1/(e^{E/T} - 1) \simeq T/E$ is the Bose-distribution function. The damping rate, γ, of a transverse soft photon is related to Γ_p as

$$\gamma = \frac{1}{4 \, n_B(E)} \, \Gamma_p \simeq \frac{E}{4 \, T} \, \Gamma_p \, . \tag{1.2}$$

In Sec. II, we compute the singular contribution to the production rate of a soft photon to leading order in real-time thermal field theory and reproduce the result in [10,11]. In Sec. III, we modify the analysis in Sec. II by substituting ${}^{\diamond}S$s for Ss, the Ss which are responsible for mass singularity, and show that mass singularity is screened. Then, the contribution to the production rate is evaluated to leading order at logarithmic accuracy, by which we mean that the factor of $O(1/\ln(g^{-1}))$ is ignored when compared to the factor of $O(1)$. The contribution thus obtained is gauge independent. In Sec. IV, we analyze corrections to the photon-quark vertex and then compute the contribution of them to the production rate. The resultant contribution coincides with the contribution obtained in Sec. III. Sec. V is devoted to discussions and conclusions. Appendix A collects formulas used in this paper. Appendix B contains calculation of ladder diagrams (cf. Fig. 6 below) for the photon–hard-quark vertex, in which some of the gluon rungs carry hard momenta. In Appendix C, we briefly analyze a class of corrections to the photon-quark vertex, which seemingly is of the same order of magnitude as the bare photon-quark vertex, and show that, eventually, it is nonleading.

We here introduce notations, $O[g^n T^\ell]$ and $O\{g^n\}$, which we use throughout this paper.

A is of $O[g^n T^\ell]$: A is of $O(g^n T^\ell)$, up to a possible factor of $\ln(g^{-1})$.

The contribution A is of $O\{g^n\}$: The contribution A is $O[g^n]$ smaller than the corresponding leading contribution.

II. LEADING-ORDER CALCULATION IN HTL-RESUMMATION SCHEME

The purpose of this section is to compute the differential rate of a soft-photon production to lowest nontrivial order within HTL-resummation scheme. We work in

massless "QCD" with the color group $SU(N_c)$ and N_f quarks.

A. Preliminary

After summing over the polarizations of the photon, the differential rate of a soft-photon production is given by

$$E\,\frac{dW}{d^3p} = \frac{i}{2\,(2\pi)^3}\,g_{\mu\nu}\,\Pi_{12}^{\mu\nu}(P)\,, \tag{2.1}$$

where $P^\mu = (E, \mathbf{p})$. In (2.1), $\Pi_{12}^{\mu\nu}$ is the $(1,2)$ component of the photon polarization tensor in the real-time formalism based on the time path $C_1 \oplus C_2 \oplus C_3$ in the complex time plane; $C_1 = -\infty \to +\infty$, $C_2 = +\infty \to -\infty$, $C_3 = -\infty \to -\infty - i/T$. [The time-path segment C_3 does not play [8,16,17] any explicit role in the present context.] The fields whose time arguments are lying on C_1 and on C_2 are referred, respectively, to as the type-1 and type-2 fields. A vertex of type-1 (type-2) fields is called a type-1 (type-2) vertex. Then $\Pi_{12}^{\mu\nu}$ in (2.1) is the "thermal vacuum polarization between the type-2 photon and the type-1 photon".

To leading order, Fig. 1 is the diagram [10,11] that contributes to $E\,dW/d^3p$. In Fig. 1, $p_0 = p = E$ and "1" and "2" stand for the thermal indices, which specify the type of vertex. Fig. 1 leads to

$$\begin{aligned}\Pi_{12}^{\mu\nu}(P) = &-i\,e_q^2 e^2 N_c \int \frac{d^4K}{(2\pi)^4}\,tr\Big[{}^*\!S_{i_1i_4}(K) \\ &\times ({}^*\Gamma^\nu(K,K'))^2_{i_4i_3}\;{}^*\!S_{i_3i_2}(K') \\ &\times ({}^*\Gamma^\mu(K',K))^1_{i_2i_1}\Big]\,,\end{aligned} \tag{2.2}$$

where $i_1, ..., i_4$ are the thermal indices that specify the field type. In (2.2), all the momenta P, K and K' are soft ($\sim gT$), so that both photon-quark vertices, ${}^*\Gamma^\nu$ and ${}^*\Gamma^\mu$, as well as both quark propagators, ${}^*\!S_{i_1i_4}$ and ${}^*\!S_{i_3i_2}$, are HTL-resummed effective ones (cf. (2.7) - (2.10) below and (A.4) - (A.8) in Appendix A). [Throughout this paper, a capital letter like P denotes the four momentum, $P = (p_0, \mathbf{p})$, and a lower-case letter like p denotes the length of the three vector, $p = |\mathbf{p}|$. The unit three vector along the direction of, say, $\mathbf{p}$ is denoted as $\hat{\mathbf{p}} \equiv \mathbf{p}/p$. The null four vector like $\hat{P}_\tau$ ($\tau = \pm$) is defined as $\hat{P}_\tau = (1, \tau\,\hat{\mathbf{p}})$ and $\hat{P} \equiv \hat{P}_{\tau=+}$.]

As a technical device, we decompose $g_{\mu\nu}$ in (2.1) into two parts as

$$g_{\mu\nu} = g^{(t)}_{\mu\nu}(\hat{\mathbf{p}}) + g^{(\ell)}_{\mu\nu}(\hat{P})\,, \tag{2.3}$$

$$g^{(t)}_{\mu\nu}(\hat{\mathbf{p}}) \equiv - \sum_{i,\,j=1}^{3} g_{\mu i}\, g_{\nu j}\, \left(\delta_{ij} - \hat{p}_i \hat{p}_j\right)\,, \tag{2.4}$$

$$g^{(\ell)}_{\mu\nu}(\hat{P}) \equiv g_{\mu 0}\hat{P}_\nu + g_{\nu 0}\hat{P}_\mu - \hat{P}_\mu \hat{P}_\nu\,. \tag{2.5}$$

Substituting (2.3) for $g_{\mu\nu}$ in (2.1), we have, with an obvious notation,

$$E\,\frac{dW}{d^3p} = E\,\frac{dW^{(t)}}{d^3p} + E\,\frac{dW^{(\ell)}}{d^3p}\,. \tag{2.6}$$

Now we observe that $({}^*\Gamma^\mu)^1_{ji}$ in (2.2) is written as

$$({}^*\Gamma^\mu(K',K))^1_{ji} = \gamma^\mu\,\delta_{1j}\,\delta_{1i} + \left({}^*\tilde{\Gamma}^\mu(K',K)\right)^1_{ji}\,, \tag{2.7}$$

$$\left({}^*\tilde{\Gamma}^\mu(K',K)\right)^1_{ji} = \frac{m_f^2}{4\pi}\int d\Omega\,\hat{Q}^\mu \hat{\not{Q}}\, f_{ji}(\hat{Q},K',K)\,, \tag{2.8}$$

where m_f, thermally induced quark mass, is defined as

$$m_f^2 = \frac{\pi\alpha_s}{2}\,C_F\,T^2 \qquad \left(C_F = \frac{N_c^2-1}{2N_c}\right) \tag{2.9}$$

and

$$f_{11} = \frac{\mathcal{P}}{K'\cdot\hat{Q}}\,\frac{\mathcal{P}}{K\cdot\hat{Q}}\,, \tag{2.10a}$$

$$f_{21} = i\pi\frac{\mathcal{P}}{K\cdot\hat{Q}}\,\delta\left(K'\cdot\hat{Q}\right) = i\pi\frac{\mathcal{P}}{P\cdot\hat{Q}}\,\delta\left(K'\cdot\hat{Q}\right)\,, \tag{2.10b}$$

$$f_{12} = -i\pi\frac{\mathcal{P}}{K'\cdot\hat{Q}}\,\delta\left(K\cdot\hat{Q}\right) = i\pi\frac{\mathcal{P}}{P\cdot\hat{Q}}\,\delta\left(K\cdot\hat{Q}\right)\,, \tag{2.10c}$$

$$f_{22} = \pi^2\,\delta\left(K'\cdot\hat{Q}\right)\,\delta\left(K\cdot\hat{Q}\right)\,. \tag{2.10d}$$

In (2.10), $\mathcal{P}$ indicates the principal-value prescription. $\left({}^*\tilde{\Gamma}^\mu\right)^1_{ji}$, Eq. (2.8), is the HTL contribution, in terms of an angular integral, and $Q^\mu = q\hat{Q}^\mu$ is the hard momentum circulating along the HTL. $({}^*\Gamma^\nu(K,K'))^2_{ji}$ in (2.2) is obtained from $({}^*\Gamma^\mu(K',K))^1_{ji}$ through the relation

$$({}^*\Gamma^\nu(K,K'))^2_{ji} = -\gamma_0\left[({}^*\Gamma^\nu(K',K))^1_{\underline{i}\underline{j}}\right]^\dagger\gamma_0\,, \tag{2.11}$$

where $\underline{i} = 2$ for $i = 1$ and $\underline{i} = 1$ for $i = 2$. Note that (2.11) is the general relation, to which the photon-quark vertex function is subjected. It turns out that the production rate $E\, dW/d^3p$ diverges due to mass singularities. As in [10,11], we are only interested in the divergent parts neglecting all finite contributions.

B. Computation of $E\, dW^{(t)}/d^3p$ in (2.6)

Mass singularities arise from the factor $1/P\cdot\hat{Q} = \{E(1-\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})\}^{-1}$ in (2.10), which diverges at $\hat{\mathbf{p}} \parallel \hat{\mathbf{q}}$. Let us see the numerator factors in the integrand of $E\, dW^{(t)}/d^3p$, which are obtained after taking the trace of Dirac matrices under the HTL approximation (cf. (2.2)).

- One of the photon-quark vertices in Fig. 1 is the HTL contribution and the other is the bare vertex.

  In $E\, dW^{(t)}/d^3p$, $\hat{Q}^\mu$ in (2.8) is to be multiplied by $g^{(t)}_{\mu\nu}(\hat{\mathbf{p}})$: $\hat{Q}^\mu g^{(t)}_{\mu\nu}(\hat{\mathbf{p}}) = g_{\nu i}[\hat{q}^i - (\hat{\mathbf{q}}\cdot\hat{\mathbf{p}})\, \hat{p}^i]$, which vanishes at $\mathbf{q} \parallel \mathbf{p}$. Then, there is no singularity in the integrand.

- Both photon-quark vertices in Fig. 1 are the HTL contributions.

  Using (2.2) and (2.8), we see that $E\, dW^{(t)}/d^3p$ includes $g^{(t)}_{\mu\nu}(\hat{\mathbf{p}})\hat{Q}^\mu\hat{Q}'^\nu = -\hat{\mathbf{q}}\cdot\hat{\mathbf{q}}' + (\hat{\mathbf{q}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}')$, where $Q'^\mu = q'\hat{Q}'^\mu$ is the hard momentum in $({}^*\Gamma^\nu)^2_{ji}$ in (2.2). For $\hat{\mathbf{q}} = \hat{\mathbf{p}}$ *and* $\hat{\mathbf{q}}' \neq \hat{\mathbf{p}}$, or for $\hat{\mathbf{q}}' = \hat{\mathbf{p}}$ *and* $\hat{\mathbf{q}} \neq \hat{\mathbf{p}}$, $g^{(t)}_{\mu\nu}(\hat{\mathbf{p}})\hat{Q}^\mu\hat{Q}'^\nu$ vanishes. For $\hat{\mathbf{q}} \simeq \hat{\mathbf{p}}$ *and* $\hat{\mathbf{q}}' \simeq \hat{\mathbf{p}}$, $g^{(t)}_{\mu\nu}(\hat{\mathbf{p}})\hat{Q}^\mu\hat{Q}'^\nu \propto (1-\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^{1/2}\ (1-\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}')^{1/2}$, and the integrations over the directions of $\hat{\mathbf{q}}$ and $\hat{\mathbf{q}}'$ converge.

Thus, $E\, dW^{(t)}/d^3p$ is free from singularity.

C. Computation of $E\, dW^{(\ell)}/d^3p$ in (2.6)

Substituting $g^{(\ell)}_{\mu\nu}(\hat{P})$, Eq. (2.5), for $g_{\mu\nu}$ in (2.1), we obtain

$$E\,\frac{dW^{(\ell)}}{d^3p} = \frac{e_q^2 e^2 N_c}{2(2\pi)^3}\left[g_{\mu 0}\hat{P}_\nu + g_{\nu 0}\hat{P}_\mu - \hat{P}_\mu\hat{P}_\nu\right]\int\frac{d^4K}{(2\pi)^4}\, tr\Big[{}^*S_{i_1 i_4}(K) \\ \times ({}^*\Gamma^\nu(K,K'))^2_{i_4 i_3}\, {}^*S_{i_3 i_2}(K')\, ({}^*\Gamma^\mu(K',K))^1_{i_2 i_1}\Big]\,. \tag{2.12}$$

It is convenient to employ here the Ward-Takahashi relations,

$$(K-K')_\mu {}^*S_{ji_2}(K')\left({}^*\Gamma^\mu(K',K)\right)^\ell_{i_2 i_1}{}^*S_{i_1 i}(K) = \delta_{\ell i}{}^*S_{ji}(K') - \delta_{\ell j}{}^*S_{ji}(K)\,. \tag{2.13}$$

On the R.H.S., no summations are taken over i and j. Using (2.13), we can easily see that the term with $\hat{P}_\mu \hat{P}_\nu$ in (2.12) does not yield mass-singular contribution. We then obtain, for the singular contributions,

$$\begin{aligned} E\,\frac{dW^{(\ell)}}{d^3p} &\simeq \frac{e_q^2 e^2 N_c}{2(2\pi)^3}\,\frac{1}{E}\int \frac{d^4K}{(2\pi)^4}\left\{tr\left[{}^*S_{2i}(K')\left({}^*\Gamma^0(K',K)\right)^1_{i2} - \left({}^*\Gamma^0(K',K)\right)^1_{2i}{}^*S_{i2}(K)\right]\right.\\ &\quad \left. + tr\left[\left({}^*\Gamma^0(K,K')\right)^2_{1i}{}^*S_{i1}(K') - {}^*S_{1i}(K)\left({}^*\Gamma^0(K,K')\right)^2_{i1}\right]\right\}\\ &= \frac{e_q^2 e^2 N_c}{(2\pi)^3}\,\frac{1}{E}\,Re\int \frac{d^4K}{(2\pi)^4}\,tr\left[{}^*S_{2i}(K')\left({}^*\Gamma^0(K',K)\right)^1_{i2}\right.\\ &\quad \left. - \left({}^*\Gamma^0(K',K)\right)^1_{2i}{}^*S_{i2}(K)\right]\,, \end{aligned} \tag{2.14}$$

where and in the following in this section the symbol "$\simeq$" is used to denote an approximation that is valid for keeping the singular contributions. The symbol "*Re*" in (2.14) means to take the real part of the quantity placed on the right of "*Re*". In obtaining (2.14), use has been made of the relation (2.11) and

$$ {}^*S_{ji}(K) = -\gamma_0\left[{}^*S_{\underline{ij}}(K)\right]^\dagger \gamma_0\,. $$

The mass-singular contributions arise from the HTL parts $\left({}^*\tilde{\Gamma}^0\right)^1_{ij}$ with $i\neq j$ of ${}^*\Gamma^0$s in (2.14). We use dimensional regularization as defined in [10], which gives

$$\int d\Omega \frac{\mathcal{P}}{P\cdot\hat{Q}} = \frac{2\pi}{E\hat{\epsilon}}\,,$$

where $\hat{\epsilon} = (D-4)/2$ with D the space-time dimension. Then, from (2.8) and (2.10), we see that the singular contributions come from the point $\mathbf{p}\parallel\mathbf{q}$:

$$\left({}^*\Gamma^\mu(K',K)\right)^1_{21} = i\frac{\pi}{2}\frac{m_f^2}{E}\frac{1}{\hat{\epsilon}}\hat{P}^\mu \hat{\not{P}}\,\delta(K'\cdot\hat{P})\,, \tag{2.15a}$$

$$\left({}^*\Gamma^\mu(K',K)\right)^1_{12} = i\frac{\pi}{2}\frac{m_f^2}{E}\frac{1}{\hat{\epsilon}}\hat{P}^\mu \hat{\not{P}}\,\delta(K\cdot\hat{P})\,. \tag{2.15b}$$

Substituting (2.15), (A.4), and (A.6) in Appendix A into (2.14), we obtain for the singular contribution,

$$\begin{aligned} E\,\frac{dW^{(\ell)}}{d^3p} &\simeq \frac{e_q^2\,e^2\,N_c}{8\pi^2}\,\frac{m_f^2}{E}\,\frac{1}{\hat{\epsilon}}\int\frac{d^4K}{(2\pi)^3}\,\delta(P\cdot K)\\ &\quad\times\sum_{\sigma=\pm}\left(\hat{K}_\sigma\cdot\hat{P}\right)\rho_\sigma(K)\,, \end{aligned} \tag{2.16}$$

where $\hat{K}_\sigma = (1, \sigma\mathbf{k})$ and the spectral function $\rho_\sigma(K)$ is defined in (A.8) in Appendix A. In deriving (2.16), use has been made of $n_F(-k_0') \simeq n_F(k_0) \simeq 1/2$, where $n_F(x) \equiv 1/(e^{x/T}+1)$ is the Fermi-distribution function. Thus, we have reproduced the result reported in [10,11].

We encounter the same integral as in (2.16) in the hard-photon production case [4]. Here we recall that K is soft $\sim O(gT)$. Then, the upper limit k^* of the integration over k is in the range, $gT << k^* << T$. Referring to [4], we have

$$E\,\frac{dW}{d^3p} \simeq \frac{e_q^2\,\alpha\,\alpha_s}{2\pi^2}\,T^2 \left(\frac{m_f}{E}\right)^2 \frac{1}{\hat{\epsilon}}\,\ln\left(\frac{k^*}{m_f}\right) . \tag{2.17}$$

The hard contribution should be added to the soft contribution (2.17). Besides a factor of $\{\ln(T/k^*) + O(1)\}$, the mechanism of arising mass singularity in the former is the same as in the latter [14]. Thus we finally obtain

$$E\,\frac{dW}{d^3p} \simeq \frac{e_q^2\,\alpha\,\alpha_s}{2\pi^2}\,T^2 \left(\frac{m_f}{E}\right)^2 \frac{1}{\hat{\epsilon}}\,\ln\left(\frac{T}{m_f}\right) .$$

III. MODIFIED HARD-QUARK PROPAGATORS AND SCREENING OF MASS-SINGULARITY

A. Preliminary

In Sec. II, we have seen that the singular contribution comes from the region $\hat{P}\cdot\hat{Q} = 1 - \hat{\mathbf{p}}\cdot\hat{\mathbf{q}} \simeq 0$ in $\left({}^*\tilde{\Gamma}^\mu\right)^1_{ji}$ with $j \neq i$ and $\hat{P}\cdot\hat{Q}' = 1 - \hat{\mathbf{p}}\cdot\hat{\mathbf{q}}' \simeq 0$ in $\left({}^*\tilde{\Gamma}^\nu\right)^2_{ji}$ with $j \neq i$. Let us first see how does the factor $\mathcal{P}/\hat{P}\cdot\hat{Q}$, which develops singularity, come about in these ${}^*\tilde{\Gamma}$s (cf. (2.8) with (2.10)).

The diagram to be analyzed is depicted in Fig. 2, where Q^μ is hard while P, K, and K' are soft. ${}^*\tilde{\Gamma}^\mu$ computed within the canonical HTL-resummation scheme is gauge independent, which diverges (cf. (2.15)). The gauge-parameter dependent part of the hard-gluon propagator (cf. (A.9) in Appendix A) leads to nonleading contribution. As mentioned in Sec. II, throughout this paper, we pursue the leading contribution that diverges and ignore finite as well as nonleading contributions. Then, we can use Feynman gauge for the gluon propagator in Fig. 2:

$$\left({}^*\tilde{\Gamma}^\mu(K',K)\right)^\ell_{ij}$$

$$= i\,(-)^{i+j+\ell} g^2\, C_F\, g^{\rho\sigma} \int \frac{d^4Q}{(2\pi)^4}\, \gamma_\rho\, S_{i\ell}(Q+K')$$

$$\times\, \gamma^\mu\, S_{\ell j}(Q+K)\, \gamma_\sigma\, \Delta_{ji}(Q)\,, \tag{3.1}$$

with no summation over ℓ, i, and j. In (3.1), $S_{\ell j}$ is the bare thermal quark propagator and $-g^{\rho\sigma}\Delta_{ji}$ is the bare thermal gluon propagator (cf. Appendix A). It is worth remarking that ${}^*\tilde{\Gamma}^\mu$s in (3.1) satisfy the relation (2.11).

Substituting (A.1) in Appendix A, we obtain for the leading contribution,

$$\left({}^*\tilde{\Gamma}^\mu(K',K)\right)^\ell_{ij}$$

$$\simeq -4i(-)^{i+j+\ell} g^2\, C_F \int \frac{d^4Q}{(2\pi)^4} \sum_{\tau=\pm} \hat{Q}^\mu_\tau$$

$$\times\, \hat{\not{Q}}_\tau\, \tilde{S}^{(\tau)}_{i\ell}(Q+K')\, \tilde{S}^{(\tau)}_{\ell j}(Q+K)\, \Delta_{ji}(Q)\,. \tag{3.2}$$

Using (A.2) and (A.3) (in Appendix A) for $\tilde{S}^{(\tau)}$s, we see that in $\left({}^*\tilde{\Gamma}^\mu\right)^1_{21}$, for example, the singular contribution comes from

$$\tilde{S}^{(\tau)}_{21}(Q+K')\, Re\tilde{S}^{(\tau)}_{11}(Q+K)$$

$$= -i\frac{\pi}{2}\, n_F(-q_0)\, \delta(q_0+k'_0-\tau|\mathbf{q}+\mathbf{k}'|)$$

$$\times\, \frac{\mathcal{P}}{q_0+k_0-\tau|\mathbf{q}+\mathbf{k}|} \tag{3.3a}$$

$$\simeq -i\frac{\pi}{2}\, n_F(-\tau q)\, \delta(q_0-\tau q+K'\cdot\hat{Q}_\tau)$$

$$\times\, \frac{\mathcal{P}}{q_0-\tau q+K'\cdot\hat{Q}_\tau} \tag{3.3b}$$

$$= -i\frac{\pi}{2}[\theta(\tau)-\tau n_F(q)]$$

$$\times\, \delta(q_0-\tau q+K'\cdot\hat{Q}_\tau)\, \frac{1}{P\cdot\hat{Q}_\tau}\,. \tag{3.3c}$$

Since $P\cdot\hat{Q}_\tau \geq 0$, the $\mathcal{P}$ prescription is dropped in the last line. For the $\tau=-$ sector, the integration variable $\mathbf{q}$ in (3.2) is changed to $-\mathbf{q}$, so that $P\cdot\hat{Q}_- \to P\cdot\hat{Q}_+ = P\cdot\hat{Q}$. Carrying out the integration over q_0 and then over $q=|\mathbf{q}|$, we obtain (2.8) with (2.10b). In (3.3c), a singularity appears at $P\cdot\hat{Q}_\tau = 0$, i.e. $\hat{\mathbf{q}} = \tau\hat{\mathbf{p}}$.

It can readily be seen that this singularity is not the artifact of the approximation made at (3.3b). In order to see this, consider the process "q" $(Q+K) \to q$ $(Q+K')$

$+\ \gamma(P)$, where $q(Q+K')$ $[\gamma(P)]$ is the on-shell quark [photon] and "q" is the off-shell quark (cf. Fig. 2). The propagator $1/(Q+K)^2$ has singularity at $\mathbf{p} \parallel (\mathbf{q}+\mathbf{k}')$. In fact, $1/(Q+K)^2 = 1/(Q+K'+P)^2 = [2\{p|\mathbf{q}+\mathbf{k}'| - \mathbf{p}\cdot(\mathbf{q}+\mathbf{k}')\}]^{-1} = \infty$, the collinear or mass singularity. As in the present example, mass singularity may emerge [18] from the small phase-space region where the momenta Rs (being kinematically constrained to $R^2 \geq 0$ or $R^2 \leq 0$) carried by bare propagators are close to the mass-shell, $R^2 \simeq 0$. Other parts of the diagram under consideration do not participate directly in the game. As seen above, relevant parts in (3.1) with $\ell = 1$, $i = 2$, and $j = 1$ are $S_{21}(Q+K')$, $S_{11}(Q+K)$, and the external photon line. We assume that the photon is not thermalized, so that no (thermal) correction to the on-shell photon should be taken into account.

Similar observation may be made for $\left({}^*\tilde{\Gamma}^\mu\right)^1_{12}$. (Note that $\left({}^*\tilde{\Gamma}^\mu\right)^1_{11}$ and $\left({}^*\tilde{\Gamma}^\mu\right)^1_{22}$ have not yielded mass-singular contribution.)

B. Modified hard-quark propagator

Above observation leads us to look into the hard-quark propagator close to the light-cone through the analysis of one-loop thermal self-energy part $\tilde{\Sigma}_F(R)$ as depicted in Fig. 3. Here $\tilde{\Sigma}_F(R)$ is the quasiparticle or diagonalized self-energy part [16]. When Q $(R-Q)$ is soft, the effective gluon (quark) propagator, ${}^*\Delta^{\rho\sigma}(Q)$ $({}^*S(R-Q))$, should be assigned to the gluon (quark) line in Fig. 3. We are aiming at constructing the $\tilde{\Sigma}_F(R)$-resummed hard-quark propagator ${}^\diamond S_F(R)$, from which ${}^\diamond S_{ij}(R)$ $(i,j=1,2)$ is obtained through standard manner [16]. When $R^2 \simeq 0$, $Im\,\tilde{\Sigma}_F(R)$ is sensitive [5–7,15] to the region where Q^2 is soft, R is hard, and $(R-Q)^2 \simeq 0$. Then, in contrast to the case of soft momentum, in determining $\tilde{\Sigma}_F(R)$ with hard R with $R^2 \simeq 0$, we need a knowledge of ${}^\diamond S_F(R-Q)$ with $(R-Q)^2 \simeq 0$. Thus, we should determine $\tilde{\Sigma}_F(R)$ in a self-consistent manner. (See, e.g., [6,15].) This is also the case for hard-gluon propagator, ${}^\diamond\Delta_F^{\mu\nu}(Q)$, and hard–FP-ghost propagator, ${}^\diamond\Delta_F^{(FP)}(Q)$.

A self-consistent determination of $\tilde{\Sigma}_F(R)$ and ${}^\diamond S(R)$s (as well as of ${}^\diamond\Delta^{\mu\nu}$s and ${}^\diamond\Delta^{(FP)}$s) is carried out in [15]. Here we summarize the result. $\tilde{\Sigma}_F(R)$ takes the form

$$\tilde{\Sigma}_F(R) \simeq \epsilon(r_0)\left[\frac{m_f^2}{r} - i\gamma_q\right]\gamma^0 + O(g^2)\times \not{R}\,, \tag{3.4}$$

where m_f is as in (2.9) and

$$\gamma_q = \frac{g^2}{4\pi} C_F T \ln(g^{-1}) \left[1 + O\left(\frac{\ln\{\ln(g^{-1})\}}{\ln(g^{-1})} \right) \right] . \tag{3.5}$$

As in [5–7,15], the form (3.5) is valid at logarithmic accuracy, i.e., the term of $O(g^2T)$ is ignored when compared to the term of $O(g^2T \ln(g^{-1}))$. It is to be noted that γ_q in (3.5) is independent of (hard-)Q. If necessary, one may explicitly evaluate the term $O(\ln(\ln(g^{-1}))/\ln(g^{-1}))$.

The (part of the) term $O(g^2) \times \not{R}$ in (3.4) is absorbed into the wave-function renormalization constant. The remainder, which depends on the renormalization scheme, gives overall correction of $O(g^2)$ to ${}^\diamond S_F(R)$ and does not affect the structure of ${}^\diamond S_F(R)$ at the region of our interest. Then, the term $O(g^2) \times \not{R}$ in (3.4) leads to the contribution of $O\{g^2\}$ to the soft-photon production rate, and we ignore it in the following. ("$O\{g^2\}$" is defined at the end of Sec. I.) The rest of the term in $\tilde{\Sigma}_F(R)$ is gauge independent.

$\tilde{\Sigma}_F(R)$-resummed propagators ${}^\diamond S_F(R)$s can be written as

$$\begin{aligned} {}^\diamond S_{j\ell}(R) &= \sum_{\tau=\pm} \hat{\not{R}}_\tau {}^\diamond\tilde{S}^{(\tau)}_{j\ell}(R) \qquad (j,\, \ell = 1,2)\,, && (3.6)\\ {}^\diamond\tilde{S}^{(\tau)}_{11}(R) &\simeq -\left[{}^\diamond\tilde{S}^{(\tau)}_{22}(R) \right]^* \\ &= \frac{1}{2} \frac{1}{r_0 - \tau\overline{r} + i\epsilon(r_0)\gamma_q} \\ &\quad + i\pi\epsilon(r_0)\, n_F(|r_0|)\, {}^\diamond\rho_\tau(R)\,, && (3.7)\\ {}^\diamond\tilde{S}^{(\tau)}_{12/21}(R) &\simeq \pm i\pi n_F(\pm r_0)\, {}^\diamond\rho_\tau(R)\,, && (3.8) \end{aligned}$$

where $\overline{r} \equiv r + m_f^2/r$ and

$$\begin{aligned} {}^\diamond\rho_\tau(R) &= \delta_{\gamma_q}[r_0 - \tau\overline{r}] \\ &\equiv \frac{1}{\pi} \frac{\gamma_q}{[r_0 - \tau\overline{r}]^2 + \gamma_q^2}\,. \end{aligned} \tag{3.9}$$

As mentioned above, the forms (3.7) - (3.9) are gauge independent.

Let us compare $\delta_{\gamma_q}(r_0 - \tau\overline{r})$ with $\delta(r_0 - \tau r)$, which is the bare counterpart of ${}^\diamond\rho_\tau(R)$ (cf. Appendix A). $\delta(r_0 - \tau r)$ "peaks" at $r_0 = \tau r$, which shifts to $r_0 = \tau(r + m_f^2/r)$ $= \tau r + O(g^2T)$ in $\delta_{\gamma_q}(r_0 - \tau\overline{r})$. The width of $\delta(r_0 - \tau r)$ is zero, while the width of $\delta_{\gamma_q}(r_0 - \tau\overline{r})$ is of $O(\gamma_q) = O(g^2T \ln(g^{-1}))$. Note that, at logarithmic accuracy, $\gamma_q >>$ $m_f^2/r = O(g^2T)$. Similar observation can be made for $Re\, {}^\diamond\tilde{S}^{(\tau)}_{11}(R)$. Thus, the free hard-quark propagator is modified in the region

$$|r_0 - \tau r| \leq O(g^2 T \, \ln(g^{-1}))\,, \tag{3.10}$$

with $\tau = \epsilon(r_0)$. More precisely [15], the forms of $Im\,{}^\diamond\tilde{S}_{11}^{(\tau)}(R)$ $(= Im\,{}^\diamond\tilde{S}_{22}^{(\tau)}(R))$ (cf. (3.7)) and ${}^\diamond\tilde{S}_{12/21}^{(\tau)}(R)$ in (3.8) are valid in the region (3.10), while the form of $Re\,{}^\diamond\tilde{S}_{11}^{(\tau)}(R)$ $(= -Re\,{}^\diamond\tilde{S}_{22}^{(\tau)}(R))$ is valid in the region $O[g^3T] < |r_0 - \epsilon(r_0)\,\overline{r}| \leq O(g^2T\,\ln(g^{-1}))$. (For the definition of "$O[g^3T]$", see the end of Sec. I.)

Comparing (A.2) and (A.3) in Appendix A with (3.7) and (3.8), we see that, in (3.3b), the following substitutions should be made,

$$\begin{aligned}
\delta(q_0 - \tau q + K' \cdot \hat{Q}_\tau) &\to {}^\diamond\rho_\tau(Q + K') \simeq \frac{1}{\pi}\,\frac{\gamma_q}{[q_0 - \tau\overline{q} + K' \cdot \hat{Q}_\tau]^2 + \gamma_q^2}\,, \\
\frac{\mathcal{P}}{q_0 - \tau q + K \cdot \hat{Q}_\tau} &\to \frac{\mathcal{P}_{\gamma_q}}{q_0 - \tau\overline{q} + K \cdot \hat{Q}_\tau} \equiv \frac{q_0 - \tau\overline{q} + K \cdot \hat{Q}_\tau}{[q_0 - \tau\overline{q} + K \cdot \hat{Q}_\tau]^2 + \gamma_q^2}\,.
\end{aligned} \tag{3.11}$$

Let $\left({}^\diamond\tilde{\Gamma}^\mu\right)_{ij}^{\ell}$ be the photon-quark vertex function that is obtained from $\left({}^*\tilde{\Gamma}^\mu\right)_{ij}^{\ell}$, Eq. (3.2), through the above replacements, which is diagrammed in Fig. 4. The substitution (3.11) results in a violation of Ward-Takahashi relation, Eq. (2.13), on the basis of which our analysis is going. This issue will be dealt with in the following section. Substituting the formulas in Appendix A into (3.2) and making the replacements (3.11), we obtain for $\left({}^\diamond\tilde{\Gamma}^\mu\right)_{21}^{1}$,

$$\begin{aligned}
&\left({}^\diamond\tilde{\Gamma}^\mu(K', K)\right)_{21}^{1} \\
&\quad \simeq 2\pi i g^2\, C_F \int \frac{d^4Q}{(2\pi)^3} \sum_{\tau=\pm} \hat{Q}_\tau^\mu\, \hat{\not{Q}}_\tau\, n_F(-q_0)[\theta(-q_0) + n_B(q)]\,\delta(Q^2) \\
&\qquad \times {}^\diamond\rho_\tau(Q + K')\,\frac{\mathcal{P}_{\gamma_q}}{q_0 - \tau\overline{q} + K \cdot \hat{Q}_\tau} \\
&\quad \simeq \frac{i}{8\pi}\, m_f^2 \sum_{\tau=\pm} \int d\Omega\, \hat{Q}_\tau^\mu\, \hat{\not{Q}}_\tau\, \frac{\gamma_q}{(K' \cdot \hat{Q}_\tau - \tau m_f^2/T)^2 + \gamma_q^2} \\
&\qquad \times \frac{K \cdot \hat{Q}_\tau - \tau m_f^2/T}{(K \cdot \hat{Q}_\tau - \tau m_f^2/T)^2 + \gamma_q^2}\,. \qquad (3.12) \\
&\quad \simeq \frac{i}{16\pi}\, m_f^2\, Im \int d\Omega\, \hat{Q}^\mu\, \hat{\not{Q}} \\
&\qquad \times \sum_{\tau=\pm} \left[\frac{1}{P \cdot \hat{Q}} \left(\frac{1}{\hat{Q} \cdot K' - \tau m_f^2/T - i\gamma_q} - \frac{1}{\hat{Q} \cdot K - \tau m_f^2/T - i\gamma_q} \right) \right. \\
&\qquad \left. - \frac{1}{P \cdot \hat{Q} - 2i\gamma_q} \left(\frac{1}{\hat{Q} \cdot K' - \tau m_f^2/T + i\gamma_q} - \frac{1}{\hat{Q} \cdot K - \tau m_f^2/T - i\gamma_q} \right) \right]. \qquad (3.13)
\end{aligned}$$

When integrating over $q = |\mathbf{q}|$ to obtain (3.12), the region $q = O(T)$ dominates, and then we have made the replacement: $\overline{q} = q + m_f^2/q \to q + m_f^2/T$.

Singling out the contribution that diverges in the limit $\gamma_q \to 0^+$, we have

$$\begin{aligned}\left({}^\diamond\tilde{\Gamma}^\mu(K', K)\right)^1_{21} \simeq \frac{i}{16} m_f^2 \int d\Omega\, \hat{Q}^\mu \hat{\not{Q}} \frac{P\cdot\hat{Q}}{(P\cdot\hat{Q})^2 + 4\gamma_q^2} \\ \times \sum_{\tau=\pm} \Big\{ \delta_{\gamma_q}(\hat{Q}\cdot K' - \tau m_f^2/T) \\ + \delta_{\gamma_q}(\hat{Q}\cdot K - \tau m_f^2/T) \Big\}\,. \end{aligned} \tag{3.14}$$

In the limit γ_q, $m_f^2/T \to 0^+$, $P\cdot\hat{Q}/\{(P\cdot\hat{Q})^2 + 4\gamma_q^2\} \to 1/P\cdot\hat{Q}$ diverges at $\mathbf{p} \parallel \mathbf{q}$, and the singular contribution (2.15a) is reproduced.

Here a comment is in order. As mentioned at the beginning of this subsection, in Fig. 4, we should assign ${}^\diamond\Delta_{ji}(Q)$ to the hard-gluon propagator (cf. (3.1)). Substitution of ${}^\diamond\Delta_{ji}(Q)$ for $\Delta_{ji}(Q)$ results in a change in δ_{γ_q}s in (3.14). However, the features stated above after in conjunction with (3.10) are unchanged, i.e., the point at which δ_{γ_q} peaks shifts by an amount of $O(g^2T)$ and the width of δ_{γ_q} is of $O(g^2T\,\ln(g^{-1}))$.

We may take the limit γ_q, $m_f^2/T \to 0^+$ in the quantity in the curly brackets in (3.14), since it does not yields any divergence at all (cf. observation in Sec. IIIA):

$$\begin{aligned}\left({}^\diamond\tilde{\Gamma}^\mu(K', K)\right)^1_{21} \simeq \frac{i}{8}\frac{m_f^2}{E} \int d\Omega\, \hat{Q}^\mu \hat{\not{Q}} \frac{1 - \hat{\mathbf{p}}\cdot\hat{\mathbf{q}}}{(1 - \hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2 + \tilde{\gamma}_q^2} \\ \times \{\delta(\hat{Q}\cdot K') + \delta(\hat{Q}\cdot K)\}\,, \end{aligned} \tag{3.15}$$

where

$$\tilde{\gamma}_q \equiv 2\gamma_q/E = O(g\,\ln(g^{-1})). \tag{3.16}$$

Thus, at logarithmic accuracy, we obtain

$$\begin{aligned}\left({}^\diamond\tilde{\Gamma}^\mu(K', K)\right)^1_{21} &\simeq \frac{i\pi}{2}\frac{m_f^2}{E} \hat{P}^\mu \hat{\not{P}}\, \delta(\hat{P}\cdot K') \\ &\quad\times \int d(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}) \frac{1 - \hat{\mathbf{p}}\cdot\hat{\mathbf{q}}}{(1 - \hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2 + \tilde{\gamma}_q^2} \\ &\simeq i\frac{\pi}{2}\frac{m_f^2}{E} \ln(1/\tilde{\gamma}_q)\, \hat{P}^\mu \hat{\not{P}}\, \delta(\hat{P}\cdot K') \\ &\simeq i\frac{\pi}{2}\frac{m_f^2}{E} \ln(g^{-1}) \hat{P}^\mu \hat{\not{P}}\, \delta(\hat{P}\cdot K')\,. \end{aligned} \tag{3.17}$$

It should be noted that this contribution comes from the region,

$$|q_i - |q_{i0}|| = O(g^2 T \, \ln(g^{-1})) \,, \qquad (i = 1, 2) \,,$$
$$O(g \, \ln(g^{-1})) \leq 1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}} << 1 \,, \tag{3.18}$$

where $Q_1 \equiv Q + K'$ and $Q_2 \equiv Q + K$ (cf. (3.10), (3.2) and (3.12)).

Through similar analysis, we obtain

$$\left({}^{\diamond}\tilde{\Gamma}^{\mu}(K', K) \right)^{1}_{12} \simeq i \frac{\pi}{2} \frac{m_f^2}{E} \, \ln(g^{-1}) \, \hat{P}^{\mu} \, \hat{\not{P}} \, \delta(K \cdot \hat{P}) \,. \tag{3.19}$$

C. Contribution to the soft-photon production rate

Comparing (3.17) and (3.19) with (2.15), we see that the replacements (3.11) make the singular contribution (2.17) the finite contribution,

$$\begin{aligned} E \frac{dW}{d^3p} &= \frac{e_q^2 e^2 N_c}{(2\pi)^3} \frac{1}{E} Re \int \frac{d^4K}{(2\pi)^4} tr \left[{}^{*}S_{2i}(K') \left({}^{\diamond}\tilde{\Gamma}^{0}(K', K) \right)^{1}_{i2} \right. \\ &\quad \left. - \left({}^{\diamond}\tilde{\Gamma}^{0}(K', K) \right)^{1}_{2i} {}^{*}S_{i2}(K) \right] \,, \\ &\simeq \frac{e_q^2 \, \alpha \, \alpha_s}{2\pi^2} T^2 \left(\frac{m_f}{E} \right)^2 \ln(g^{-1}) \ln \left(\frac{k^*}{m_f} \right) \,, \end{aligned} \tag{3.20}$$

which is valid at logarithmic accuracy. As has been already mentioned, the result (3.20) is gauge independent.

As to the hard contribution, the mass singularity is cutoff [14] in the same way as in the soft contribution analyzed above, which produces a $\ln(g^{-1})$ as in (3.20). Adding the hard contribution to (3.20), we obtain

$$\begin{aligned} E \frac{dW}{d^3p} &\simeq \frac{e_q^2 \, \alpha \, \alpha_s}{2\pi^2} T^2 \left(\frac{m_f}{E} \right)^2 \ln(g^{-1}) \ln \left(\frac{T}{m_f} \right) \\ &\simeq \frac{e_q^2 \, \alpha \, \alpha_s}{2\pi^2} T^2 \left(\frac{m_f}{E} \right)^2 \ln^2(g^{-1}) \,. \end{aligned} \tag{3.21}$$

In computing dW/d^3p, Eq. (3.20), for soft-quark propagators, we have used ${}^{*}S$s, which is evaluated in canonical HTL-resummation scheme. However, to be consistent, in computing the soft-quark self-energy part, Fig. 5, ${}^{\diamond}S$ and ${}^{\diamond}\Delta$ should be assigned, in respective order, to the quark- and gluon-lines in the HTL. ${}^{*}S(K)$ is written in terms

of $D_\sigma(K)$ ($\sigma = \pm$), Eq. (A.7) in Appendix A. Let us briefly see how does the form of $D_\sigma(K)$ change by the above-mentioned replacements, $S \to {}^\diamond S$ and $\Delta \to {}^\diamond\Delta$. In $D_\sigma(K)$, the term $\sigma m_f^2/k$ is insensitive to the region $Q^2 \simeq 0$ and/or $(K-Q)^2 \simeq 0$ in Fig. 5. Then, the change in the term $\sigma m_f^2/k$ is of higher order. The logarithmic factor in $D_\sigma(K)$, $\ln[\{k_0(1+0^+)+k\}/\{k_0(1+0^+)-k\}]$, changes to

$$\ln\left(\frac{k_0+k+a+i\epsilon(k_0)\,b}{k_0-k+a+i\epsilon(k_0)\,b}\right),$$

where $|a| = O(g^2T)$ and $b = O(g^2T\,\ln(g^{-1}))$. Therefore, the change in the logarithmic factor is appreciable only in the region, $|k-|k_0|| \leq O[g^2T]$, which is small when compared to the whole soft (k_0,k)-region ($|k_0| \leq k$). Then the leading-order contribution (3.20) is not affected by this change (cf. (2.16) and [4]).

IV. PHOTON-QUARK VERTEX CORRECTIONS

A. Preliminary

Deducing the result (3.21) is not the end of the analysis. In deriving (2.14) or (3.20), we have used the Ward-Takahashi relation (2.13), which is a representation of the current-conservation condition or the gauge invariance. Substitution of ${}^\diamond\tilde{\Gamma}^\mu$ (cf. (3.12)) for ${}^*\tilde{\Gamma}^\mu$ in (2.13) violates the current-conservation condition.

For recovering it, one needs to include corrections to the photon-quark vertex in Fig. 4. Here we face the question: In the kinematical region of our interest, Eq. (3.18), what kind of diagrams does participate. In other wards, what kind of vertex corrections leads to the contribution, which is of the same order of magnitude as the bare photon-quark vertex.

Similar problem arises in the damping rate of a moving (quasi)particle in a hot QCD/QED plasma. Lebedev and Smilga [7] have shown that the relevant diagrams are the ladder diagrams as depicted in Fig. 6, where all the gluon rungs carry the soft momenta. (In Fig. 6, solid- and dashed-lines stand, respectively, for quark- and gluon-propagators, P is soft, and Q_1 (and then also Q_2 $(= Q_1 + P)$ is hard). The region of our interest is (cf. (3.18))

$$|q_{j0} - \epsilon(q_{j0})q_j| = O(\gamma_q) = O(g^2T\,\ln(g^{-1}))\,,$$

$$(j=1,2)\,, \tag{4.1}$$

$$E-\tau\hat{\mathbf{q}}_1\cdot\mathbf{p}=O(\gamma_q)\,, \tag{4.2}$$

where γ_q is as in (3.5). *At logarithmic accuracy*, the leading contribution comes [7] from the magnetic sector of the effective gluon propagators. That some other diagrams than Fig. 6 lead to nonleading contributions are discussed in [7,15].

The contribution from Fig. 6 with n-rungs reads [7]

$$\left(\hat{\Lambda}^\mu_n(Q_1,Q_2)\right)^\ell_{ji}\simeq-(-)^{i+j+\ell}\gamma^0\,\hat{Q}^\mu_{1\tau}\sum_{\rho=\pm}\left[\mathcal{N}^{(\rho)}_{j\ell}\mathcal{N}^{(-\rho)}_{\ell i}\left\{\frac{-2i\rho\tau\gamma_q}{E-\tau\hat{\mathbf{q}}_1\cdot\mathbf{p}-2i\rho\tau\gamma_q}\right\}^n\right]\,, \tag{4.3}$$

where the sum is not taken over ℓ, j, and i, $Q_2-Q_1=P$, and

$$\begin{aligned}\mathcal{N}^{(+)}_{11}&=-\mathcal{N}^{(-)}_{22}=1-n_F(q_1)\\ \mathcal{N}^{(-)}_{11}&=-\mathcal{N}^{(+)}_{22}=n_F(q_1)\\ \mathcal{N}^{(+)}_{12/21}&=-\mathcal{N}^{(-)}_{12/21}=\theta(\mp q_{10})-n_F(q_1)\,.\end{aligned} \tag{4.4}$$

$\hat{\Lambda}^\mu_n$ in (4.3) meets (2.11), which serves as a cross-check of the validity of (4.3).

It is straightforward to resum $\hat{\Lambda}^\mu_n(Q_1,Q_2)$ over n:

$$\left(\hat{\Lambda}^\mu(Q_1,Q_2)\right)^\ell_{ji}\equiv\sum_{n=1}^\infty\left[\hat{\Lambda}^\mu_n(Q_1,Q_2)\right]^\ell_{ji}\,,\qquad(\ell=1,2)\,, \tag{4.5a}$$

$$\left(\hat{\Lambda}^\mu(Q_1,Q_2)\right)^\ell_{11}\simeq\left(\hat{\Lambda}^\mu(Q_1,Q_2)\right)^\ell_{22}\simeq0\,, \tag{4.5b}$$

$$\left(\hat{\Lambda}^\mu(Q_1,Q_2)\right)^\ell_{12/21}\simeq\pm2i\,\tau(-)^\ell\,\gamma^0\,\hat{Q}^\mu_{1\tau}[\theta(\mp q_{10})-n_F(q_1)]\frac{\gamma_q}{E-\tau\hat{\mathbf{q}}_1\cdot\mathbf{p}}\,, \tag{4.5c}$$

where $\tau=\epsilon(q_{10})$. Thus, through resummation, the imaginary part of the denominator in $\hat{\Lambda}^\mu_{n=1}(Q_1,Q_2)$, Eq. (4.3), "disappears", which is the important finding in [7]. [Kraemmer, Rebhan, and Schultz [13] have discussed that the same phenomenon takes place in scalar QCD.]

It should be emphasized that the result (4.5c) is valid to leading order *at logarithmic accuracy*, i.e., valid in the region, (4.1) and (4.2). More precisely, γ_q in

the numerator of (4.3) and γ_q in the denominator is the same only at logarithmic accuracy, $\gamma_q = O(g^2 T \ln(g^{-1}))$.

It is straightforward to show [7] that, to the accuracy we are taking, the photon-quark vertex function

$$(\Lambda^\mu(Q_1, Q_2))^1_{ji} \equiv \delta_{1i}\, \delta_{1j}\, \gamma^\mu + \left(\hat{\Lambda}^\mu(Q_1, Q_2)\right)^1_{ji} \tag{4.6}$$

satisfies the Ward-Takahashi relation,

$$P_\mu \, (\Lambda^\mu(Q_1, Q_2))^1_{ji} \simeq \delta_{1j} {}^\diamond S^{-1}_{1i}(Q_2) - \delta_{1i} {}^\diamond S^{-1}_{j1}(Q_1) \,, \tag{4.7}$$

(with $P = Q_2 - Q_1$). ${}^\diamond S^{-1}_{ji}(Q)$ is the inverse-matrix function of ${}^\diamond S_{ji}(Q)$:

$$\begin{aligned} {}^\diamond S^{-1}_{11}(Q) &= - \left[{}^\diamond S^{-1}_{22}(Q) \right]^* \\ &= \not{Q} - \tilde{\Sigma}_F(Q) + 2 i n_F(q) \, Im \tilde{\Sigma}_F(Q) && \text{(4.8a)} \\ &\simeq \not{Q} - \tau \, \gamma^0 [m_f^2/q - i\{1 - 2n_F(q)\} \, \gamma_q] \,, \\ & && \text{(4.8b)} \\ {}^\diamond S^{-1}_{12/21}(Q) &= 2i[\theta(\mp q_0) - n_F(q)] \, Im \tilde{\Sigma}_F(Q) && \text{(4.8c)} \\ &\simeq -2i\tau \, \gamma^0 [\theta(\mp q_0) - n_F(q)] \, \gamma_q \,. && \text{(4.8d)} \end{aligned}$$

As a matter of fact, using (4.5) and (4.8), we see that the difference between the L.H.S. of (4.7) and the R.H.S. is of $O[g^3 T]$.

B. Estimate of the form for $\hat{\Lambda}^\mu(Q_1, Q_2)$

For the purpose of inferring the form of $\hat{\Lambda}^\mu(Q_1, Q_2)$ that holds in much wider region than (4.2), we reverse the order of argument. Namely, after *imposing* Ward-Takahashi relation (4.7), diagrammatic analysis follows. The region of our interest here is

$$\Delta T \equiv |E - \tau \hat{\mathbf{q}}_1 \cdot \mathbf{p}| \leq O(g^2 T \, \ln(g^{-1})) \,. \tag{4.9}$$

It is sufficient to analyze $\left(\hat{\Lambda}^\mu\right)^1_{ij}$, since $\left(\hat{\Lambda}^\mu\right)^2_{ij}$ is obtained from $\left(\hat{\Lambda}^\mu\right)^1_{ij}$ through (2.11).

We first make preliminary remarks. Since $|\tilde{\Sigma}_F(Q_1)| = O[g^2 T]$ and $|\tilde{\Sigma}_F(Q_2) - \tilde{\Sigma}_F(Q_1)| = O[g^3 T]$, (4.7) with (4.8) yields

$$\left|P_\mu \left(\hat\Lambda^\mu\right)^1_{11/22}\right| = O[g^3 T]\,, \qquad \left|P_\mu \left(\hat\Lambda^\mu\right)^1_{12/21}\right| = O[g^2 T]\,. \tag{4.10}$$

This fact, together with (4.5b), indicates that, for our purpose, $\left(\hat\Lambda^\mu\right)^1_{11/22}$ may be ignored. Then we concentrate our concerns on $\left(\hat\Lambda^\mu\right)^1_{12/21}$. Eq. (4.10) together with $|P^\mu|$ $= O(gT)$ also shows that it is sufficient to analyze the structure of $\left(\hat\Lambda^\mu\right)^1_{12/21}$ up to and including $O[g]$ contribution, with the proviso to be discussed below. Furthermore, as in Secs. II and III and as will be shown below, we need only $\left(\hat\Lambda^{\mu=0}\right)^1_{12/21}$ *to leading order*, provided that $\left(\hat\Lambda^\mu\right)^1_{12/21}$ satisfies the Ward-Takahashi relation. Whenever confusion does not arise in the following, we drop thermal indices. The last remark is on the 4×4 matrix structure of $\hat\Lambda^\mu(Q_1, Q_2)$. We recall that in computing the modified HTL contribution to the photon-quark vertex, $\hat{\not{Q}}_{1\tau}$ ($\hat{\not{Q}}_{2\tau}$) is to be multiplied from the left (right) of $\hat\Lambda^{\mu=0}(Q_1, Q_2)$. Then the term in $\hat\Lambda^{\mu=0}(Q_1, Q_2)$ that is proportional to $\hat{\not{Q}}_{1\tau} \simeq \hat{\not{Q}}_{2\tau}$ may be ignored. This is because $\left(\hat{\not{Q}}_{1\tau}\right)^2 = 0$ and, from (4.9), $|\hat{\not{Q}}_{1\tau}\,\hat{\not{Q}}_{2\tau}|$ is at most of $O[g^{3/2}]$.

1. Resummation of ladder diagrams

With the above preliminary remarks in mind, we start with the analysis of (resummation of) ladder diagrams, Fig. 6, which yields [7] the leading contribution. The region, where momenta of the quark lines adjacent to each gluon rung are soft, $|Q^{(j)\,\mu}| = O(gT) = |R^{(j)\,\mu}|$, is unimportant, because the phase-space volume is small. The contribution from the region, $\left|\left(Q^{(j)}\right)^2\right| = O(T^2) = \left|\left(R^{(j)}\right)^2\right|$, is of $O\{g^2\}$. ("$O\{g^2\}$" is defined at the end of Sec. I.) The leading contribution comes from the region $\left|\left(R^{(j)}\right)^2\right|, \left|\left(Q^{(j)}\right)^2\right| \leq O(\gamma_q T) = O[g^2 T]$ ($j = 2, ..., n+1$). Then it is sufficient to assume that each K_j ($j = 1, ..., n$) is either soft or hard with nearly (anti)collinear to Q_1^μ ($\simeq Q_2^\mu$).

In Appendix B, we show that the contribution of Fig. 6, where at least one K_j out of $K_2 - K_n$ is hard, is of $O[g^2]$. The contribution of Fig. 6 with all Ks but K_1 are soft is of the form

$$\hat\Lambda^\mu(Q_1, Q_2) = \mathcal{F}_1^\mu(Q_1, Q_2)\,\hat{\not{Q}}_{1\tau} + \mathcal{F}_2^\mu(Q_1, Q_2)\,\hat{\not{Q}}_{2\tau} \\ + O\{g^2\}\,.$$

Thus, according to the preliminary remarks above, we can ignore this contribution.

Let us turn to analyze Fig. 6 with all Ks soft. We pick out the term

$$\hat{\not{Q}}_\tau \, \gamma^\mu \, \hat{\not{R}}_\tau \left(\equiv \hat{\not{Q}}_\tau^{(n+1)} \, \gamma^\mu \, \hat{\not{R}}_\tau^{(n+1)} \right) ,$$

with $\tau = \epsilon(q_{10}) = \epsilon(q_{20})$. Noting that $\left(\hat{\not{Q}}_\tau\right)^2 = 0$, we obtain

$$\hat{\not{Q}}_\tau \, \gamma^\mu \, \hat{\not{R}}_\tau = 2\hat{Q}_\tau^\mu \, \hat{\not{R}}_\tau + \tau \, \gamma^\mu \, \hat{\not{Q}}_\tau \, \frac{1}{q} \left[\vec{\gamma} \cdot \{ \mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{q}}) \, \hat{\mathbf{q}} \} \right] . \tag{4.11}$$

The first term on the R.H.S. of (4.11)

We first study the contribution, ${}^{(1)}\hat{\Lambda}^\mu(Q_1, Q_2)$, arising from the first term on the R.H.S. For ${}^{(1)}\hat{\Lambda}^{\mu=0}(Q_1, Q_2)$, the first term reads $2\hat{Q}_\tau^{\mu=0} \, \hat{\not{R}}_\tau = 2\hat{\not{R}}_\tau$.

For studying $P_\mu {}^{(1)}\hat{\Lambda}^\mu$, we begin with

$$\begin{aligned} P \cdot \hat{Q}_\tau = P \cdot \hat{Q}_{1\tau} &- \frac{\tau}{q_1} (\mathbf{p} \cdot \underline{\mathbf{k}}_\perp) \\ &+ \frac{\tau}{2q_1^2} \left[2(\hat{\mathbf{q}}_1 \cdot \underline{\mathbf{k}})(\mathbf{p} \cdot \underline{\mathbf{k}}_\perp) + (\underline{\mathbf{k}}_\perp)^2 (\mathbf{p} \cdot \hat{\mathbf{q}}_1) \right] \\ &+ O(g^4) , \end{aligned} \tag{4.12}$$

where $\underline{\mathbf{k}}_\perp \equiv \underline{\mathbf{k}} - (\underline{\mathbf{k}} \cdot \hat{\mathbf{q}}_1) \, \hat{\mathbf{q}}_1$ with

$$\underline{\mathbf{k}} \equiv \sum_{j=1}^{n} \mathbf{k}_j , \tag{4.13}$$

which is soft. The last term on the R.H.S. of (4.12) is of $O(g^3 T)$ and, according to the preliminary remarks above, it seems to be ignored. However, (4.9) tells us that the first term on the R.H.S. is of $O[\Delta T]$ with $\Delta \leq O[g^2]$. Then, for the purpose of finding the limit (of Δ) of validity of the form $\hat{\Lambda}^\mu$ obtained below, we keep the last term in question.

From the second term on the R.H.S. of (4.12), we take up $\mathbf{p} \cdot \mathbf{k}_\perp / q_1 = p \, k \, \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}_\perp / q_1$ $(K^\mu \equiv K_n^\mu)$ and trace $\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}_\perp$, which appears in the integral

$$\langle \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}_\perp \rangle \equiv \int d\Omega_{\hat{\mathbf{k}}} \, \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}_\perp \, {}^\diamond\tilde{S}^{(\tau)}(Q) \, {}^\diamond\tilde{S}^{(\tau)}(R) . \tag{4.14}$$

Here $d\Omega_{\hat{\mathbf{k}}}$ stands for the integration over the direction of $\hat{\mathbf{k}}$. ${}^{\diamond}\tilde{S}^{(\tau)}(Q)$ may be written as (cf. (3.7) - (3.9))

$$
\begin{aligned}
{}^{\diamond}\tilde{S}^{(\tau)}(Q) &= \frac{1}{2}\sum_{\rho=\pm}\frac{\mathcal{N}^{(\rho)}(Q)}{q_0-\tau\overline{q}+i\rho\tau\Gamma(q)} && (4.15)\\
&\simeq \frac{1}{2}\sum_{\rho=\pm}\frac{\mathcal{N}^{(\rho)}(Q^{(n)})}{q_0^{(n)}-\tau\overline{q}^{(n)}+K\cdot\hat{Q}_\tau^{(n)}+i\rho\tau\Gamma(q^{(n)})}\,, && (4.16)
\end{aligned}
$$

where $\mathcal{N}^{(\rho)}$s are as in (4.4) and $\Gamma(q)=\gamma_q+O(g^2T)$ with γ_q as in (3.5). Similarly,

$$
\begin{aligned}
{}^{\diamond}\tilde{S}^{(\tau)}(R) &\simeq \frac{1}{2}\sum_{\sigma=\pm}\frac{\mathcal{N}^{(\sigma)}(R)}{r_0^{(n)}-\tau\overline{r}^{(n)}+K\cdot\hat{R}_\tau^{(n)}+i\sigma\tau\Gamma(r)}\\
&\simeq \frac{1}{2}\sum_{\sigma=\pm}\frac{\mathcal{N}^{(\sigma)}(Q^{(n)})}{r_0^{(n)}-\tau\overline{r}^{(n)}+K\cdot\hat{Q}_\tau^{(n)}-\tau\mathbf{k}\cdot\mathbf{p}_T/q^{(n)}+i\sigma\tau\Gamma(q^{(n)})}\,, && (4.17)
\end{aligned}
$$

where $\mathbf{p}_T\equiv\mathbf{p}-(\mathbf{p}\cdot\hat{\mathbf{q}}^{(n)})\,\hat{\mathbf{q}}^{(n)}$. The dominant contribution comes from where the denominators of (4.16) and (4.17) are of $O[g^2T]$. To the approximation we are keeping in mind,

$$
\hat{\mathbf{k}}_\perp\simeq\hat{\mathbf{k}}_T=\hat{\mathbf{k}}-(\hat{\mathbf{k}}\cdot\hat{\mathbf{q}}^{(n)})\,\hat{\mathbf{q}}^{(n)}\,.
$$

We take the direction $\hat{\mathbf{q}}^{(n)}$ as the z-axis and the direction $\mathbf{p}_T$ as the x-axis. Then $\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}_\perp\simeq\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}_T=O[(\Delta/g)^{1/2}]\cos\phi$ with ϕ the azimuth. ${}^{\diamond}\tilde{S}^{(\tau)}(Q)$ in (4.16) is independent of ϕ. In ${}^{\diamond}\tilde{S}^{(\tau)}(R)$, only term that depends on ϕ is $-\tau\mathbf{k}\cdot\mathbf{p}_T/q^{(n)}$:

$$
\begin{aligned}
-\tau\,\frac{\mathbf{k}\cdot\mathbf{p}_T}{q^{(n)}} &= -\tau\frac{k_T\,p_T}{q^{(n)}}\,\cos\phi\\
&= O[g^{3/2}\Delta^{1/2}T]\le O[g^{5/2}T]\,.
\end{aligned}
$$

Then, the relative order of magnitude of $-\tau\,\mathbf{k}\cdot\mathbf{p}_T/q^{(n)}$ in the denominator of (4.17) is of $O[g^{3/2}\Delta^{1/2}T]/O[g^2T]=O[(\Delta/g)^{1/2}]$.

After all this, we find

$$
\frac{\langle\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}_\perp\rangle}{\langle 1\rangle}=O[\Delta/g]\,,
$$

or

$$
\frac{\langle\mathbf{p}\cdot\mathbf{k}_\perp/q_1\rangle}{\langle 1\rangle}=O[g\Delta T]\,, \qquad (4.18)
$$

where $\langle 1 \rangle$ is defined by (4.14) with $\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}_\perp$ deleted. It is not difficult to see that undoing the approximation, $\hat{\mathbf{k}}_\perp \simeq \hat{\mathbf{k}}_T$, used above, affects the term of $O[g^2 \Delta T]$ in (4.18).

Similar analysis goes for $\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}_{j\perp}$ $(1 \leq j \leq n-1)$ in the second term on the R.H.S. of (4.12) and the same conclusion as above results.

From the above analysis, we see that $P \cdot \hat{Q}_\tau$ in (4.12) turns out to

$$P \cdot \hat{Q}_\tau = E - \tau \mathbf{p} \cdot \hat{\mathbf{q}}_1 \left[1 - \frac{1}{2q_1^2} (\underline{\mathbf{k}}_\perp)^2 \right] + \text{“}O\text{”}[g\Delta T] \,, \tag{4.19}$$

where “O”$[g\Delta T]$ means the term that leads to $O\{g\Delta T\}$ contribution to $P_\mu \hat{\Lambda}^\mu$. We note that $E - \tau \mathbf{p} \cdot \hat{\mathbf{q}}_1 = O[\Delta T]$, Eq. (4.9), and then the term “O”$[g\Delta T]$ may be ignored. As mentioned above, the term $\tau \mathbf{p} \cdot \hat{\mathbf{q}}_1 (\underline{\mathbf{k}}_\perp)^2/(2q_1^2)$ in (4.19) is of $O[g^3]$ and, according to the preliminary remarks above after (4.9), seems to be ignored. It cannot be overemphasized, however, that *this term necessarily appears* in the combination *as in (4.19)* and we keep it.

From the analysis made so far, we may write ${}^{(1)}\hat{\Lambda}_n^{\mu=0}$, the contribution from Fig. 6 with n rungs, and $P_\mu {}^{(1)}\hat{\Lambda}_n^\mu$ as,

$$
\begin{aligned}
&{}^{(1)}\hat{\Lambda}_n^{\mu=0}(Q_1, Q_2) \simeq \mathcal{G}_n(Q_1, Q_2) \,, && (4.20)\\
&P_\mu {}^{(1)}\hat{\Lambda}_n^\mu(Q_1, Q_2) \\
&\qquad \simeq [E - \tau \mathbf{p} \cdot \hat{\mathbf{q}}_1 (1 - f_n)] \, \mathcal{G}_n(Q_1, Q_2) + O\{g^3\} \,. && (4.21)
\end{aligned}
$$

Here f_n has come from $(\underline{\mathbf{k}}_\perp)^2/(2q_1^2)$ in (4.19), *which is positive for all* n and is of $O(g^2)$. Then, thanks to the first mean value theorem, we can safely assume that f_n $(n = 1, 2, ...)$ is of $O(g^2)$ and (at least $Re\, f_n$) is positive. Now we choose $f \equiv f_j$ such that, for all i $(\neq j)$, $Re\, f_j \leq Re\, f_i$. Eq. (4.19) with (4.13) indicates that presumably $f_j = f_1$. Then, summing over n, we obtain

$$
\begin{aligned}
{}^{(1)}\hat{\Lambda}^{\mu=0}(Q_1, Q_2) &\equiv \sum_{n=1}^{\infty} {}^{(1)}\hat{\Lambda}_n^{\mu=0}(Q_1, Q_2) \\
&\simeq \mathcal{G}(Q_1, Q_2) \,, && (4.22)\\
P_\mu {}^{(1)}\hat{\Lambda}^\mu(Q_1, Q_2) &\equiv P_\mu \sum_{n=1}^{\infty} {}^{(1)}\hat{\Lambda}_n^\mu(Q_1, Q_2) \\
&\simeq [E - \tau \mathbf{p} \cdot \hat{\mathbf{q}}_1 (1 - f)] \, \mathcal{G}(Q_1, Q_2) + O\{g^3\} \,. && (4.23)
\end{aligned}
$$

In obtaining (4.23), the term $\tau \mathbf{p} \cdot \hat{\mathbf{q}}_1 (f_n - f) \, \mathcal{G}_n$ in $P_\mu {}^{(1)}\hat{\Lambda}_n^\mu$ has been absorbed into the term $O\{g^3\}$ in (4.21) and then in (4.23). f may depends on the (dropped) thermal

indices. However, this dependence does not become an obstacle for our purpose (cf. next subsection).

The second term on the R.H.S. of (4.11)

Let us turn to analyze the contribution, ${}^{(2)}\hat{\Lambda}^\mu(Q_1, Q_2)$, which arises from the second term on the R.H.S. of (4.11):

$$\mathcal{E}^\mu \equiv \tau \, \gamma^\mu \, \hat{\not{Q}}_\tau \, \frac{1}{q} \, [\vec{\gamma} \cdot \{\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{q}}) \, \hat{\mathbf{q}}\}] \; . \tag{4.24}$$

This is of $O(g)$ and does not contribute to $\hat{\Lambda}^{\mu=0}(Q_1, Q_2)$ to leading order. Then it is sufficient to analyze the contribution to Ward-Takahashi relation. Multiplying $P_\mu (= Q_{2\mu} - Q_{1\mu})$ to $\mathcal{E}^\mu$ and summing over μ, we obtain

$$\begin{aligned}
&\tau \not{P} \, \hat{\not{Q}}_{1\tau} \, \frac{1}{q_1} \, [\vec{\gamma} \cdot \{\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{q}}_1) \, \hat{\mathbf{q}}_1\}] \\
&\quad \simeq \tau \left\{ \gamma^0 [E - \tau (q_2 - q_1)] + \tau \, q_2 \, \hat{\not{Q}}_{2\tau} \right\} \hat{\not{Q}}_{1\tau} \, \frac{1}{q_1} \vec{\gamma} \cdot \mathbf{p}_\perp \\
&\qquad + O[g^{5/2}] \\
&\quad \simeq \tau \left[\gamma^0 (E - \tau \mathbf{p} \cdot \hat{\mathbf{q}}_1) - \vec{\gamma} \cdot \mathbf{p}_\perp \right] \hat{\not{Q}}_{1\tau} \, \frac{1}{q_1} \vec{\gamma} \cdot \mathbf{p}_\perp \\
&\qquad + O[g^{5/2}] \\
&\quad = O[g \Delta T] \times \hat{\not{Q}}_{1\tau} + O[g^{5/2}] \, .
\end{aligned}$$

This is of the same order of magnitude as $O[g\Delta T]$ in (4.19) and we can ignore this contribution.

2. Nonladder diagram

Finally we make a brief analysis of nonladder diagram, Fig. 7, the contribution of which is of $O\{g\}$ [7]. Then, as in the case of (4.24), it is sufficient to analyze the contribution to Ward-Takahashi relation. As above, the dominant contribution comes from $|R^2|$, $|Q^2| = O[g^2 T^2]$, where $|\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{q}}) \, \hat{\mathbf{q}}| = O[(g\Delta)^{1/2} T] \leq O[g^{3/2} T]$. We pick out the term,

$$P_\mu \, \hat{\not{Q}}_\xi \gamma^\mu \, \hat{\not{R}}_\xi = 2 (P \cdot \hat{Q}_\xi) \, \hat{\not{R}}_\xi + ... \, , \tag{4.25}$$

where $\xi = \epsilon(q_0) = \epsilon(r_0)$. In (4.25), "..." leads to the contribution (to $P_\mu \hat{\Lambda}^\mu(Q_1, Q_2)$), which is absorbed into the term $O\{g^3\}$ in (4.23) and may be ignored. The first term leads to

$$P_\mu \, \hat{\Lambda}^\mu(Q_1, Q_2) = \left(E - \tau \mathbf{p} \cdot \hat{\mathbf{q}}_1 + O[g^3] \right) \mathcal{G}'(Q_1, Q_2) \, .$$

Then $\mathcal{G}'(= O\{g\})$ here can be absorbed into $\mathcal{G}$ in (4.23).

3. The form for $\hat{\Lambda}^\mu(Q_1, Q_2)$

Through the qualitative analysis made above, we have learnt that the structures of $\hat{\Lambda}^{\mu=0}(Q_1, Q_2)$ (to leading order) and $P_\mu \hat{\Lambda}^\mu(Q_1, Q_2)$ are given by (4.22) and (4.23), respectively:

$$\hat{\Lambda}^{\mu=0}(Q_1, Q_2) \simeq \mathcal{G}(Q_1, Q_2) \, , \tag{4.26a}$$

$$P_\mu \, \hat{\Lambda}^\mu(Q_1, Q_2) \simeq [E - \tau \mathbf{p} \cdot \hat{\mathbf{q}}_1 (1 - f_n)] \, \mathcal{G}(Q_1, Q_2) + O\{g^3\} \, . \tag{4.26b}$$

We have to emphasize that the forms (4.26) should not be taken too seriously. This is because (4.26) is not the "calculated" result but is obtained by assuming the Ward-Takahashi relation supplemented with diagrammatic analysis. Nevertheless, to go further, we assume the forms (4.26a) and (4.26b) in the following.

To (generalized) one-loop order, $\tilde{\Sigma}(Q)$ may be decomposed as

$$\tilde{\Sigma}(Q) \simeq \gamma^0 \, \mathcal{H}_0(Q) + \hat{\not{Q}}_\tau \, \mathcal{H}_v(Q) \, , \tag{4.27}$$

where $q_0 \simeq \tau q$. Then, from (4.7), (4.8) with (4.27), we see that, $\mathcal{G}(Q_1, Q_2)$ in (4.26) may be decomposed as

$$\mathcal{G}(Q_1, Q_2) \simeq \gamma^0 \, \mathcal{G}_0(Q_1, Q_2) + \hat{\not{Q}}_{1\tau} \, \mathcal{G}_v(Q_1, Q_2) \, .$$

According to the preliminary remarks above after (4.9), the second term on the R.H.S. is not important for our purpose.

Now we are ready to determine $[\mathcal{G}_0(Q_1, Q_2)]^1_{12/21}$ or $[\hat{\Lambda}^{\mu=0}(Q_1, Q_2)]^1_{12/21}$, to leading order. By substituting (4.26b) into (4.7) with (4.8d), we find

$$\left(\hat{\Lambda}^{\mu=0}(Q_1,Q_2)\right)^1_{12/21} \simeq \mp 2i\tau\,\gamma^0[\theta(\mp q_{10}) - n_F(q_1)] \times \frac{\gamma_q}{E - \tau \mathbf{p}\cdot\hat{\mathbf{q}}_1(1-f)} . \tag{4.28}$$

It should be emphasized that $Re\, f > 0$ and $|f| = O[g^2]$. Then (4.28) is not singular at $|\mathbf{p}\cdot\hat{\mathbf{q}}_1|$. In (4.28), we have set $\Gamma(q_1) = \gamma_q$, being valid at logarithmic accuracy and gauge independent, which is sufficient for our purpose. Also to be emphasized is the fact that the R.H.S. of (4.7) with $j \neq i$ (and then also $\left(\hat{\Lambda}^{\mu=0}\right)^1_{j\neq i}$) is independent of $Re\,\tilde{\Sigma}_F(Q)$ $(= \epsilon(q_0)(m_f^2/q)\gamma^0)$.

It is worth mentioning that, in determining the form of $\hat{\Lambda}^{\mu=0}$, Eq. (4.28), through the qualitative analysis, we have not used the explicit form of the soft-gluon propagator. Then, the "qualitative" result (4.28) does not depend on the HTL-resummed (approximate) form for the soft-gluon propagator.

C. Contribution to the modified effective photon-quark vertex

We are now in a position to compute Fig. 8, where the photon-quark vertex with the square blob indicates $\hat{\Lambda}^\mu$, the zeroth component of which is given by (4.28). The original effective photon-quark vertex, Fig. 2, is gauge independent and we are dealing with the modification of it near the light-cone, $(Q+K)^2 \simeq (Q+K')^2 \simeq 0$. Then, as in Secs. II and III, we use the Feynman gauge for the hard-gluon propagator in Fig. 8:

$$\begin{aligned}
&\left({}^\diamond\hat{\Gamma}^\mu(K',K)\right)^1_{ji} \\
&\quad = -i(-)^{i+j} g^2\, C_F \int \frac{d^4Q}{(2\pi)^4}\, \gamma^\rho\, {}^\diamond S_{j\ell}(Q_1) \\
&\qquad \times \left[\hat{\Lambda}^\mu(Q_1,Q_2)\right]^1_{\ell k} {}^\diamond S_{ki}(Q_2)\, \gamma_\rho\, \Delta_{ij}(Q)\,,
\end{aligned} \tag{4.29}$$

where $Q_1 = Q + K'$ and $Q_2 = Q + K$. As in Secs. II and III and as will be seen below, we need only $\left({}^\diamond\hat{\Gamma}^{\mu=0}\right)^1_{ji}$ with $i \neq j$, to leading order at logarithmic accuracy.

As in Sec. IIIB, for hard gluon line with Q in Fig. 8, we are allowed to use the bare propagator, $\Delta_{ij}(Q)$ (cf. (4.29)). Straightforward manipulation using (4.28) yields

$$
\begin{aligned}
\left({}^{\diamond}\hat{\Gamma}^{\mu=0}(K',K)\right)^{1}_{12} \simeq & -\frac{ig^2}{2}\,C_F\sum_{\tau=\pm}\tau\int\frac{d^3q_1}{(2\pi)^3}\,\frac{1}{q_1}\,\hat{\not{Q}}_{1\tau} \\
& \times[\theta(\tau)+n_B(q_1)][\theta(-\tau)-n_F(q_1)]\frac{\tilde{\gamma}_q}{1-\tau\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}_1(1-f)} \\
& \times\sum_{\rho=\pm}\left[\frac{\rho\{\theta(\rho)-n_F(q_1)\}}{\hat{Q}_{1\tau}\cdot K'-\tau m_f^2/q_1+i\rho\tau\gamma_q}\right. \\
& \left.\times\frac{1}{\hat{Q}_{1\tau}\cdot K-\tau m_f^2/q_1-i\rho\tau\gamma_q}\right],
\end{aligned}
\tag{4.30}
$$

where $\tilde{\gamma}_q$ $(=O[g])$ is as in (3.16). Let us pick out

$$
\begin{aligned}
& \int_{-1}^{1}dz\,\frac{\tilde{\gamma}_q}{1-\tau\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}_1(1-f)} \\
& \quad\times\sum_{\rho=\pm}\left[\frac{\rho\{\theta(\rho)-n_F(q_1)\}}{\left(\hat{Q}_{1\tau}\cdot K'-\tau m_f^2/q_1+i\rho\tau\gamma_q\right)\left(\hat{Q}_{1\tau}\cdot K-\tau m_f^2/q_1-i\rho\tau\gamma_q\right)}\right],
\end{aligned}
\tag{4.31}
$$

where $z\equiv-\tau\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}$. We proceed as in Sec. III (cf. (3.14) - (3.17)). Rewrite (4.31) as

$$
\begin{aligned}
\text{Eq. (4.31)} = & \frac{\tilde{\gamma}_q}{E}\int_{-1}^{1}\frac{dz}{1+z(1-f)}\sum_{\rho=\pm}\left[\frac{\rho\{\theta(\rho)-n_F(q_1)\}}{1+z-i\rho\tau\tilde{\gamma}_q}\right. \\
& \left.\times\left\{\frac{1}{\hat{Q}_{1\tau}\cdot K'-\tau m_f^2/q_1+i\rho\tau\gamma_q}-\frac{1}{\hat{Q}_{1\tau}\cdot K-\tau m_f^2/q_1-i\rho\tau\gamma_q}\right\}\right].
\end{aligned}
\tag{4.32}
$$

The contribution of our interest comes from $z\simeq-1$ or $\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}_1\simeq\tau$. Then $\hat{Q}_{1\tau}\cdot K'\simeq\hat{Q}_{1\tau}\cdot K$. Thus we have

$$
\begin{aligned}
\text{Eq. (4.32)} \simeq & \frac{2\pi}{E}\,\delta_{\gamma_q}(K\cdot\hat{P}-\tau m_f^2/q_1) \\
& \times\left[\{1-n_F(q_1)\}\ln\left(\frac{-i\tau\tilde{\gamma}_q}{f}\right)\right. \\
& \left.+n_F(q_1)\ln\left(\frac{i\tau\tilde{\gamma}_q}{f}\right)\right].
\end{aligned}
\tag{4.33}
$$

After all this, as in Sec. III, we may set $\delta_{\gamma_q}(K\cdot\hat{P}-\tau m_f^2/q_1)\to\delta(K\cdot\hat{P})$ and, at logarithmic accuracy, we get

$$
\begin{aligned}
\text{Eq. (4.33)} & \simeq 2\pi\,\ln\left(\frac{\tilde{\gamma}_q}{|f|}\right)\,\delta(K\cdot P) \\
& \simeq 2\pi\,\ln(g^{-1})\,\delta(K\cdot P).
\end{aligned}
\tag{4.34}
$$

Substituting (4.34) into (4.30), we obtain

$$\begin{aligned}\left({}^{\diamond}\hat{\Gamma}^{\mu=0}(K', K)\right)^1_{12} &\simeq \frac{i\pi}{2}\, m_f^2 \,\ln(g^{-1})\, \hat{\not{P}}\, \delta(K \cdot P) \\ &= \left({}^{\diamond}\tilde{\Gamma}^{\mu=0}(K', K)\right)^1_{12} \\ &= \left({}^{\diamond}\tilde{\Gamma}^{\mu=0}(K', K)\right)^1_{21} \,. \end{aligned} \tag{4.35}$$

Similar analysis yields

$$\left({}^{\diamond}\hat{\Gamma}^{\mu=0}(K', K)\right)^1_{21} \simeq \left({}^{\diamond}\hat{\Gamma}^{\mu=0}(K', K)\right)^1_{12} \,. \tag{4.36}$$

D. Contribution to the soft-photon production rate

The contribution (to the soft-photon production rate) of our concern is (cf. (2.12))

$$\begin{aligned} E\,\frac{dW^{(\ell)}}{d^3p} &= \frac{e_q^2 e^2 N_c}{2(2\pi)^3}\left[g_{\mu 0}\hat{P}_\nu + g_{\nu 0}\hat{P}_\mu - \hat{P}_\mu \hat{P}_\nu\right] \\ &\quad \times \int \frac{d^4K}{(2\pi)^4}\, tr\Big[{}^{*}S_{i_1 i_4}(K)\, ({}^{\diamond}\Gamma^\nu(K', K))^2_{i_4 i_3}\ {}^{*}S_{i_3 i_2}(K')\, ({}^{\diamond}\Gamma^\mu(K', K))^1_{i_2 i_1}\Big]\,, \end{aligned} \tag{4.37}$$

where

$$({}^{\diamond}\Gamma^\mu)^\ell_{jk} = -(-)^\ell \delta_{\ell j}\, \delta_{\ell k}\, \gamma^\mu + \left({}^{\diamond}\tilde{\Gamma}^\mu\right)^\ell_{jk} + \left({}^{\diamond}\hat{\Gamma}^\mu\right)^\ell_{jk} \,. \tag{4.38}$$

${}^{\diamond}\tilde{\Gamma}^\mu$ is the contribution from Fig. 4, which has been dealt with in Sec. IIIB (cf. (3.17) and (3.19)), while ${}^{\diamond}\hat{\Gamma}^\mu$ is the contribution from Fig. 8. ${}^{\diamond}\tilde{\Gamma}^\mu + {}^{\diamond}\hat{\Gamma}^\mu$ is given by (4.29), where $\hat{\Lambda}^\mu(Q_1, Q_2)$ is replaced by $\Lambda^\mu(Q_1, Q_2)$, Eq. (4.6). Then, (4.37) includes the contribution (3.20) obtained in Sec. III.

Using (4.7) with (4.6), we find

$$\begin{aligned} &{}^{*}S_{j i_2}(K')\, (P_\mu\, {}^{\diamond}\Gamma^\mu(K', K))^\ell_{i_2 i_1}\ {}^{*}S_{i_1 i}(K) \\ &\quad \simeq \delta_{\ell i}\, {}^{*}S_{ji}(K') - \delta_{\ell j}\, {}^{*}S_{ji}(K), \end{aligned} \tag{4.39}$$

with no summation over i and j. Here we have used the fact that, for our purpose, we can set ${}^{\diamond}S_{ji}(K) \to {}^{*}S_{ji}(K)$ (cf. observation made at the end of Sec. III).

Using (4.39) and (4.38), we obtain

$$E\,\frac{dW^{(\ell)}}{d^3p} \simeq \frac{2e_q^2e^2N_c}{(2\pi)^3}\,\frac{1}{E}\,Re\int\frac{d^4K}{(2\pi)^4}\,tr\left[{}^*S_{2i}(K')\left({}^\diamond\tilde\Gamma^0(K',K)\right)^1_{i2}\right.$$
$$\left. -\left({}^\diamond\tilde\Gamma^0(K',K)\right)^1_{2i}\,{}^*S_{i2}(K)\right]\,, \tag{4.40}$$

where use has been made of (4.35) and (4.36). Note that the first term on the R.H.S. of (4.38), when substituted into (4.40), does not yield the leading contribution *at logarithmic accuracy.* Comparison of (4.40) with (2.14), with ${}^*\Gamma^0(K',K) \to {}^\diamond\tilde\Gamma^0(K',K)$, shows that $E\,dW^{(\ell)}/d^3p$ in (4.40) is twice as large as (3.20).

Adding the hard contribution to (4.40), we finally obtain

$$E\,\frac{dW}{d^3p} \simeq \frac{e_q^2\,\alpha\,\alpha_s}{\pi^2}\,T^2\left(\frac{m_f}{E}\right)^2\,\ln(g^{-1})\ln\left(\frac{T}{m_f}\right)$$
$$\simeq \frac{e_q^2\,\alpha\,\alpha_s}{\pi^2}\,T^2\left(\frac{m_f}{E}\right)^2\,\ln^2(g^{-1})\,, \tag{4.41}$$

which is valid at logarithmic accuracy and is gauge independent. Half of EdW/d^3p above comes from the region $O[g] \leq 1-\hat{\mathbf{p}}\cdot\hat{\mathbf{q}} << O(1)$ in ${}^\diamond\tilde\Gamma^{\mu=0}(K',K)$ in (3.15). The remaining half comes from the region $O[g^2] \leq 1-\tau\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}_1 \leq O[g]$ in ${}^\diamond\hat\Gamma^{\mu=0}(K',K)$ in (4.30) with (4.31). This is the central result of this paper.

There is one comment to make about the usage of ${}^*S$s in (4.37). Just as in the photon-quark vertex dealt with in this section, corrections to the quark-gluon vertices in Fig. 5 should be taken into account. From the analysis in this section, it can readily be recognized that the corrections are important only in the region, $|k-|k_0|| \leq O[g^2T]$. Then, the same observation as above, made at the end of Sec. III, applies and the leading-order result (4.41) holds unchanged.

Seemingly a correction to the photon-quark vertex as depicted in Fig. 9, where Q_1 and Q_2 are hard and all the three gluon lines carry soft momenta, is of $O(1)$, the same order of magnitude as the bare photon-quark vertex. In appendix C, we show that, as a matter of fact, the correction, Fig. 9, is at most of $O[g]$, so that it does not lead to leading contribution to the soft-photon production rate. It is straightforward to extend the analysis in Appendix C to more general diagrams for the photon-quark vertex, in which many soft-gluon lines participate. We then find that they do not yield the leading contribution to $\hat\Lambda^\mu(Q_1,Q_2)$.

V. DISCUSSIONS AND CONCLUSIONS

In Sec. II, within the canonical HTL-resummation scheme, we have analyzed the diagram that lead to logarithmically divergent leading contribution to the soft-photon production rate. The diverging factor $1/\hat{\epsilon}$ comes from mass singularity. As has been pointed out in [12–14], if the calculation of some quantity within the HTL-resummation scheme results in a diverging result, it is a signal of necessity of further resummation. In Sec. III, by replacing the hard-quark propagators Ss with ${}^{\diamond}S$s, which is obtained by resumming one-loop self-energy part $\tilde{\Sigma}_F$ in a self-consistent manner, we have shown that the mass singularity is screened and the diverging factor $1/\hat{\epsilon}$ in the production rate turns out to $\ln(g^{-1})$. Replacement $S \to {}^{\diamond}S$ violates the current-conservation condition, to which the photon-quark vertex subjects. In Sec. IV, we have estimated corrections to the photon-quark vertex, which inevitably comes in for restoring the current-conservation condition. The estimated corrections yield the leading contribution to the soft-photon production rate, which coincides with the contribution deduced in Sec. III.

Thus, we have obtained for the soft-photon production rate,

$$E\,\frac{dW}{d^3p} \simeq \frac{e_q^2\,\alpha\,\alpha_s}{\pi^2}\,T^2\left(\frac{m_f}{E}\right)^2\,\ln^2(g^{-1})\,, \tag{5.1}$$

which is valid at logarithmic accuracy and is gauge independent. The result (5.1) is twice as large as the result reported in [14], in which the "asymptotic masses" are resummed for hard propagators. In the present case, the asymptotic mass m_f is in the denominators of ${}^{\diamond}\tilde{S}$s, Eqs. (3.7) - (3.9), the mass which comes from the real part of the (hard) quark self-energy part, $Re\,\tilde{\Sigma}_F(R) \simeq \tau(m_f^2/r)\gamma^0$, Eq. (3.4). However, at logarithmic accuracy, the term that should be kept in ${}^{\diamond}S$s is not $-\tau m_f^2/r$ but the term including γ_q, the "damping rate", which comes from $Im\,\tilde{\Sigma}_F(R)$.

Much interest has been devoted to the damping rate of moving quanta in a hot plasma [1,5–7]. Using (1.1), (1.2), and (5.1), we obtain for the damping rate γ of a soft photon in a quark-gluon plasma,

$$\gamma \simeq 2\pi e_q^2\,\alpha\,\alpha_s\,T\left(\frac{m_f}{E}\right)^2\,\ln^2(g^{-1})\,.$$

Finally, we make general observation on the structure of a generic thermal reaction rate. Consider a generic formally higher-order diagram contributing to the reaction

rate. Obviously, its contribution is really of higher order, in so far as that the loop-momentum integrations are carried out over the "hard phase-space" region. Here the "hard phase-space region" is the region where all the loop momenta Rs are not only hard but also are "hard", $|R^\mu| >> O(gT)$ and $R^2 >> O((gT)^2)$. Then, the only possible source of emerging leading contributions from such diagrams are the infrared region and/or the region close to the light-cone in the loop-momentum space.

As has already been mentioned at the end of Sec. IIIA, the mechanism of arising mass singularities due to hard propagators are the same as in vacuum theory [18,19], since the statistical factor is finite for hard momentum. A mass singularity may arise from a collinear configuration of massless particles. Other parts of the diagram do not participate directly in the game. Then, as far as the mass-singular contributions (that turns out to "$\ln(g^{-1})$ contribution") are concerned, we can use bare propagators for all but the hard lines that are responsible for mass singularities. In case of soft-photon production rate, dealt with in this paper, the hard gluon line in Figs. 2, 4, and 8 and hard lines constituting the HTL of the soft-quark self-energy part in Fig. 1 are such propagators. After all, we see that using $^\diamond S$, $^\diamond\Delta^{\mu\nu}$, and $^\diamond\Delta^{(FP)}$, for, in respective order, the relevant hard quark, gluon, and FP-ghost propagators in the diverging (formally) higher-order diagram render the diverging contribution finite.

As to the contribution from the infrared region or from the region where infrared region and the region close to the light-cone overlap each other, we cannot draw any definite conclusion at present. In fact, the structure of a generic thermal amplitude in such regions remains to be elucidated, the issue which is still under way [12–14]. We like to stress here that this issue is *not* inherent in the present case of the soft-photon production. In fact, any thermal reaction rate shares the same problem, even if it is finite to leading order.

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APPENDIX A THERMAL PROPAGATORS

Here we display various expressions and useful formulas, which are directly used in this paper.

Bare thermal propagator of a quark

$$S_{j\ell}(Q) = \sum_{\tau=\pm} \hat{\not{Q}}_\tau \tilde{S}^{(\tau)}_{j\ell}(Q)\,, \tag{A.1}$$

$$\begin{aligned}
\tilde{S}^{(\tau)}_{11}(Q) &= -\left\{\tilde{S}^{(\tau)}_{22}(Q)\right\}^*\,,\\
&= \frac{1}{2\{q_0(1+i0^+) - \tau q\}}\\
&\quad + i\pi\epsilon(q_0)n_F(q)\,\delta(q_0 - \tau q)\,,
\end{aligned} \tag{A.2}$$

$$\tilde{S}^{(\tau)}_{12/21}(Q) = \pm i\pi n_F(\pm q_0)\,\delta(q_0 - \tau q)\,. \tag{A.3}$$

Effective thermal propagator of a soft quark

$$^*S_{ji}(K) = \sum_{\sigma=\pm} \hat{\not{K}}_\sigma\, {}^*\tilde{S}^{(\sigma)}_{ji}(K)\,, \qquad (j, i = 1, 2)\,, \tag{A.4}$$

where

$$\begin{aligned}
\hat{K}_\sigma &= (1,\, \sigma\hat{\mathbf{k}})\,,\\
{}^*\tilde{S}^{(\sigma)}_{11}(K) &= -\left({}^*\tilde{S}^{(\sigma)}_{22}(K)\right)^*\\
&= -\frac{1}{2D_\sigma(k_0(1+i0^+), k)}\\
&\quad + i\pi\,\epsilon(k_0)n_F(|k_0|)\rho_\sigma(K)\,,
\end{aligned} \tag{A.5}$$

$$^*\tilde{S}^{(\sigma)}_{12/21}(K) = \pm i\pi\, n_F(\pm k_0)\,\rho_\sigma(K)\,, \tag{A.6}$$

with

$$\begin{aligned}
D_\sigma(K) &= -k_0 + \sigma k\\
&\quad + \frac{m_f^2}{2k}\left[\left(1 - \sigma\frac{k_0}{k}\right)\ln\frac{k_0 + k}{k_0 - k} \,+\, 2\sigma\right]\,,
\end{aligned} \tag{A.7}$$

$$\rho_\sigma(K) = \frac{\epsilon(k_0)}{2\pi i}\left[\frac{1}{D_\sigma(k_0(1+i0^+), k)}\right.$$

$$
-\frac{1}{D_\sigma(k_0(1-i0^+),k)}\Bigg] \,, \qquad \text{(A.8)}
$$

$$
m_f^2 = \frac{\pi\alpha_s}{2}\,\frac{N_c^2-1}{2N_c}\,T^2 \,.
$$

$D_\sigma(K)$ in (A.7) is first calculated in [20].

Bare thermal gluon propagator

$$
\Delta_{j\ell}^{\mu\nu}(Q) = -\left[g^{\mu\nu} - \eta\, Q^\mu\, Q^\nu \frac{\partial}{\partial\lambda^2}\right]\, \Delta_{j\ell}(Q;\lambda^2)\Big|_{\lambda\,=\,0} \,,
$$

$$
(j,\,\ell = 1,2) \,, \qquad \text{(A.9)}
$$

$$
\Delta_{11}(Q;\lambda^2) = -\left\{\Delta_{22}(Q;\lambda^2)\right\}^*
$$

$$
= \frac{1}{Q^2-\lambda^2+i0^+} - 2\pi i n_B(|q_0|)\,\delta(Q^2-\lambda^2) \,, \qquad \text{(A.10)}
$$

$$
\Delta_{12/21}(Q;\lambda^2)
$$

$$
= -\,2\pi i\left[\theta(\mp q_0) + n_B(|q_0|)\right]\,\delta(Q^2-\lambda^2) \,. \qquad \text{(A.11)}
$$

APPENDIX B LADDER DIAGRAM WITH HAD GLUON EXCHANGE

In this Appendix, we shall show that Fig. 6 with hard K_j $(2 \le j \le n)$ yields $O\{g^2\}$ contribution to $\hat{\Lambda}^\mu(Q_1,Q_2)$, while the contribution from hard K_1 is of the form $O(1) \times \hat{\not{Q}}_{1\tau} \simeq O(1) \times \hat{\not{Q}}_{2\tau}$. The analysis here is of "minimum" in the sense that we stop the analysis when we obtain the results that are sufficient for our purpose.

Let K_j in Fig. 6 be hard. From $\left|\left(Q^{(j)}\right)^2\right|, \left|\left(R^{(j)}\right)^2\right| << T^2$ (cf. Sec. IVB), we can show that the important configurations are as follows:

(a) $\epsilon(k_j^0) = \epsilon_j$:

$$\begin{aligned}
&\mathbf{k}_j \text{ and } \mathbf{q}^{(j)} \text{ are nearly parallel,} \\
&\epsilon_{j+1} = \epsilon_j \, , \\
&K_j^\mu / k_j \simeq Q^{(j)\,\mu}/q^{(j)} \\
&\qquad\quad \simeq Q^{(j+1)\,\mu}/q^{(j+1)} \, .
\end{aligned}$$

(b) $\epsilon(k_j^0) = -\epsilon_j$:

$$\begin{aligned}
&\mathbf{k}_j \text{ and } \mathbf{q}^{(j)} \text{ are nearly antiparallel,} \\
&\epsilon_{j+1} = \epsilon_j \, \epsilon(q^{(j)} - k_j) \, , \\
&K_j^\mu / k_j \simeq -Q^{(j)\,\mu}/q^{(j)} \\
&\qquad\quad \simeq -\epsilon(q^{(j)} - k_j) \, Q^{(j+1)\,\mu}/q^{(j+1)} \, ,
\end{aligned}$$

where $\epsilon_j \equiv \epsilon(q_0^{(j)})$. Taking this into account, we obtain

$$q^{(j+1)} \simeq |q^{(j)} + \zeta k_j| - \frac{\zeta q^{(j)} k_j (1 - \zeta \hat{\mathbf{q}}^{(j)} \cdot \hat{\mathbf{k}}_j)}{|q^{(j)} + \zeta k_j|} \, , \tag{B.1}$$

$$\hat{Q}^{(j+1)\,\mu}_{\epsilon_{j+1}} \simeq \hat{Q}^{(j)\,\mu}_{\epsilon_j} + \epsilon_{j+1} \left(0, \, \underline{\mathbf{q}}^{(j)} \right)^\mu \, , \tag{B.2}$$

$$\begin{aligned}
\underline{\mathbf{q}}^{(j)} = \; & \frac{\mathbf{k}_j - \zeta k_j \hat{\mathbf{q}}^{(j)}}{|q^{(j)} + \zeta k_j|} \\
& + \frac{\epsilon_{j+1} \, \epsilon(k_j^0) \, q^{(j)} \, k_j (1 - \zeta \hat{\mathbf{q}}^{(j)} \cdot \hat{\mathbf{k}}_j)}{|q^{(j)} + \zeta k_j|^2} \, \hat{\mathbf{q}}^{(j)} \, ,
\end{aligned} \tag{B.3}$$

where $\zeta \equiv \epsilon_j \, \epsilon(k_j^0)$ and $\hat{Q}^{(j)\,\mu}_{\epsilon_j} \simeq \hat{Q}^\mu_{1\tau}$ or $-\hat{Q}^\mu_{1\tau}$.

The quark propagator ${}^\diamond\tilde{S}^{(\epsilon_{j+1})}(Q^{(j+1)})$, Eq. (4.15), may be written as

$${}^\diamond\tilde{S}^{(\epsilon_{j+1})}(Q^{(j+1)}) \simeq \frac{1}{2} \sum_{\rho = \pm} \frac{\mathcal{N}^{(\rho)}(Q^{(j+1)})}{\mathcal{D}_\rho} \, , \tag{B.4}$$

where

$$\begin{aligned}
\mathcal{D}_\rho &= q_0^{(j+1)} - \epsilon_{j+1} \overline{q}^{(j+1)} + i\rho\epsilon_{j+1} \, \gamma_q(q^{(j+1)}) \\
&\simeq q_0^{(j)} + k_j^0 - \epsilon_{j+1} |q^{(j)} + \zeta k_j| - \epsilon_{j+1} \, \frac{m_f^2}{|q^{(j)} + \zeta k_j|} \\
&\quad + \epsilon(k_j^0) \, \frac{q^{(j)} \, k_j}{q^{(j)} + \zeta k_j} \, (1 - \zeta \hat{\mathbf{q}}^{(j)} \cdot \hat{\mathbf{k}}_j) \\
&\quad + i\rho \, \epsilon_{j+1} \gamma_q(q^{(j+1)}) \, .
\end{aligned} \tag{B.5}$$

As has been mentioned in Sec. IVB, the leading contribution to $\hat{\Lambda}^\mu(Q_1, Q_2)$ comes from the region; $\left| \left(Q^{(j)} \right)^2 \right|$, $\left| \left(Q^{(j+1)} \right)^2 \right|$, $|K_j^2| \leq O[g^2 T^2]$. Then from (B.4) and (B.5), we see that the important region is

$$\begin{aligned}
&1-\zeta\,\hat{\mathbf{q}}^{(j)}\cdot\hat{\mathbf{k}}_j\,, \quad 1-\zeta\,\hat{\mathbf{r}}^{(j)}\cdot\hat{\mathbf{k}}_j \le O[g^2]\,, \\
&|k_j^0-\epsilon(k_j^0)\,k_j| \le O[g^2T]\,.
\end{aligned} \tag{B.6}$$

The region, e.g., $1-\zeta\,\hat{\mathbf{q}}^{(j)}\cdot\hat{\mathbf{k}}_j = O[g]$ yields $O\{g\}$ contribution to $\hat{\Lambda}^\mu(Q_1,Q_2)$.

The hard-gluon propagator ${}^\diamond\Delta^{\rho\sigma}(K)$ consists [15] of three part, the transverse part, the longitudinal part, and the gauge part. Their Lorentz-tensor structure are $\mathcal{P}_T^{\rho\sigma}(\hat{\mathbf{k}}) \equiv -\sum_{i,\,j=1}^3 g^{\rho i}g^{\sigma j}(\delta^{ij}-\hat{k}^i\hat{k}^j)$, $\mathcal{P}_L^{\rho\sigma}(K)\equiv g^{\rho\sigma}-K^\rho K^\sigma/K^2-\mathcal{P}_T^{\rho\sigma}(\hat{\mathbf{k}})$, and $K^\rho K^\sigma/K^2$, respectively.

Now, from Fig. 6, we pick out

$$\begin{aligned}
\mathcal{I}_Q^{\sigma_j} &\equiv \gamma^{\sigma_j}\,\hat{\not{Q}}_{\epsilon_{j+1}}^{(j+1)} \\
&= -\hat{\not{Q}}_{\epsilon_{j+1}}^{(j+1)}\,\gamma^{\sigma_j} + 2\hat{Q}_{\epsilon_{j+1}}^{(j+1)\,\sigma_j} \\
&\equiv \mathcal{I}_Q^{(1)\,\sigma_j} + \mathcal{I}_Q^{(2)\,\sigma_j}\,, \\
\mathcal{I}_R^{\rho_j} &\equiv \hat{\not{R}}_{\epsilon_{j+1}}^{(j+1)}\,\gamma^{\rho_j}\,, \\
&= -\gamma^{\rho_j}\,\hat{\not{R}}_{\epsilon_{j+1}}^{(j+1)} + 2\hat{R}_{\epsilon_{j+1}}^{(j+1)\,\rho_j} \\
&\equiv \mathcal{I}_R^{(1)\,\rho_j} + \mathcal{I}_R^{(2)\,\rho_j}\,.
\end{aligned} \tag{B.7}$$

1. $\mathcal{I}_Q^{(1)}\otimes\mathcal{I}_R^{(1)}$

We start with analyzing $\mathcal{I}_Q^{(1)\,\sigma_j}\otimes\mathcal{I}_R^{(1)\,\rho_j}$. Using (B.2) and (B.3), we obtain

$$\mathcal{I}_Q^{(1)\,\sigma_j} = -\left[\hat{\not{Q}}_{\epsilon_j}^{(j)} - \epsilon_{j+1}\vec{\gamma}\cdot\underline{\mathbf{q}}^{(j)}\right]\gamma^{\sigma_j}\,, \tag{B.8}$$

$$\mathcal{I}_R^{(1)\,\rho_j} = -\gamma^{\rho_j}\left[\hat{\not{R}}_{\epsilon_j}^{(j)} - \epsilon_{j+1}\vec{\gamma}\cdot\underline{\mathbf{r}}^{(j)}\right]\,, \tag{B.9}$$

where $\underline{\mathbf{q}}^{(j)}$ is as in (B.3) and is of $O[g]$ in the region (B.6). $\underline{\mathbf{r}}^{(j)}$ is defined by (B.3) with $Q^{(j)}\to R^{(j)}$.

For $2\le j\le n$, $\hat{\not{Q}}_{\epsilon_j}^{(j)}$ $\left[\hat{\not{R}}_{\epsilon_j}^{(j)}\right]$ is to be multiplied from the left [right] of (B.8) [(B.9)] and then the first terms in the square brackets in (B.8) and (B.9) vanish. Then,

$$\hat{\not{Q}}_{\epsilon_j}^{(j)}\,\mathcal{I}_Q^{(1)\,\sigma_j}\otimes\mathcal{I}_R^{(1)\,\rho_j}\,\hat{\not{R}}_{\epsilon_j}^{(j)} \le O[g^2]\,.$$

According to the preliminary remarks in Sec. IVB, this contribution can be ignored.

For $j=1$, we have

$$\mathcal{I}_Q^{(1)\,\sigma_1} \otimes \mathcal{I}_R^{(1)\,\rho_1} = \hat{\not{Q}}_{1\tau}\,\gamma^{\sigma_1} \otimes \gamma^{\rho_1}\,\hat{\not{Q}}_{2\tau} \\ -\epsilon_2\{\vec{\gamma}\cdot\underline{\mathbf{q}}^{(1)}\}\gamma^{\sigma_1}\otimes\gamma^{\rho_1}\,\hat{\not{Q}}_{2\tau} \\ -\epsilon_2\hat{\not{Q}}_{1\tau}\,\gamma^{\sigma_1}\otimes\gamma^{\rho_1}\,\{\vec{\gamma}\cdot\underline{\mathbf{r}}^{(1)}\} + O[g^2]\,. \tag{B.10}$$

To be consistent with the Ward-Takahashi relation (4.7), the 4×4 matrix structure of $P_\mu\hat{\Lambda}^\mu(Q_1,Q_2)$ should be of the form (cf. (4.27)),

$$P_\mu\hat{\Lambda}^\mu(Q_1,Q_2) = \gamma^0\,\mathcal{F}_0(Q_1,Q_2) + \hat{\not{Q}}_{1\tau}\,\mathcal{F}_1(Q_1,Q_2) \\ +\hat{\not{Q}}_{2\tau}\,\mathcal{F}_2(Q_1,Q_2)\,. \tag{B.11}$$

Comparison of (B.10) and (B.11) tells us that the contribution from $\mathcal{I}_Q^{(1)\,\sigma_1}\otimes\mathcal{I}_R^{(1)\,\rho_1}$ to $\hat{\Lambda}^\mu(Q_1,Q_2)$ is of the form

$$\hat{\Lambda}^\mu(Q_1,Q_2) = O(1)\times\hat{\not{Q}}_{1\tau} + O(1)\times\hat{\not{Q}}_{2\tau} + O[g^2]\,.$$

Here the first two contributions on the R.H.S. come from Fig. 6, where all Ks but K_1 are soft.

2. $\mathcal{I}_Q^{(1)}\otimes\mathcal{I}_R^{(2)}$ *and* $\mathcal{I}_Q^{(2)}\otimes\mathcal{I}_R^{(1)}$

Let us turn to analyze $\mathcal{I}_Q^{(1)\,\sigma_j}\otimes\mathcal{I}_R^{(2)\,\rho_j}$. The first entry to be analyzed is

$$\mathcal{I}_Q^{(1)\,\sigma_j}\otimes\mathcal{I}_R^{(2)\,\rho_j}\,\mathcal{P}_{T\,\rho_j\sigma_j}(\hat{\mathbf{k}}_j) \ni \left[\hat{\not{Q}}^{(j)}_{\epsilon_j} - \epsilon_{j+1}\vec{\gamma}\cdot\underline{\mathbf{q}}^{(j)}\right]\gamma^{\sigma_j}\,\hat{R}^{(j+1)\,\rho_j}_{\epsilon_{j+1}}\,\mathcal{P}_{T\,\rho_j\sigma_j}(\hat{\mathbf{k}}_j) \\ = -\epsilon_{j+1}\left[\hat{\not{Q}}^{(j)}_{\epsilon_j} - \epsilon_{j+1}\vec{\gamma}\cdot\underline{\mathbf{q}}^{(j)}\right] \\ \times\left[\vec{\gamma}\cdot\hat{\mathbf{r}}^{(j+1)} - (\vec{\gamma}\cdot\hat{\mathbf{k}}_j)(\hat{\mathbf{k}}_j\cdot\hat{\mathbf{r}}^{(j+1)})\right]\,. \tag{B.12}$$

We see from (B.6) with $j\to j+1$ that the quantity in the second curly brackets in (B.12) is of $O[g]$. Then, for $2\le j\le n$, we have

$$\hat{\not{Q}}^{(j)}_{\epsilon_j}\times\text{(B.12)} \le O[g^2]$$

and, for $j=1$,

$$(\text{B.12}) \leq \hat{\not{Q}}_{1\tau} \times O[g] + O[g^2]\,.$$

The second entry is

$$
\begin{aligned}
&\mathcal{I}_Q^{(1)\,\sigma_j} \otimes \mathcal{I}_R^{(2)\,\rho_j}\; g_{\rho_j\sigma_j} \\
&\qquad \ni \left[\hat{\not{Q}}^{(j)}_{\epsilon_j} - \epsilon_{j+1}\vec{\gamma}\cdot\underline{\mathbf{q}}^{(j)}\right]\,\gamma^{\sigma_j}\,\hat{R}^{(j+1)\,\rho_j}_{\epsilon_{j+1}}\; g_{\rho_j\sigma_j} \\
&\qquad = \left[\hat{\not{Q}}^{(j)}_{\epsilon_j} - \epsilon_{j+1}\vec{\gamma}\cdot\underline{\mathbf{q}}^{(j)}\right]\,\hat{\not{R}}^{(j+1)}_{\epsilon_{j+1}}\,.
\end{aligned}
\tag{B.13}
$$

From $\mathbf{r}^{(j+1)} = \mathbf{q}^{(j+1)} + \mathbf{p}$, we obtain

$$\hat{\mathbf{r}}^{(j+1)} \simeq \hat{\mathbf{q}}^{(j+1)} + \frac{1}{q^{(j+1)}}\left[\mathbf{p} - (\mathbf{p}\cdot\hat{\mathbf{q}}^{(j+1)})\,\hat{\mathbf{q}}^{(j+1)}\right]\,. \tag{B.14}$$

In the region (4.9) and (B.6), the second term on the R.H.S. is of $O[(g\Delta)^{1/2}] \leq O[g^{3/2}]$. Then, using (B.2) with (B.3) and $(\hat{Q}^{(j)}_{\epsilon_j})^2 = 0$, we see that, in (B.13), $\hat{\not{Q}}^{(j)}_{\epsilon_j}\,\hat{\not{R}}^{(j+1)}_{\epsilon_{j+1}} = \hat{\not{Q}}^{(j)}_{\epsilon_j} \times O[g]$. The remainder of (B.13) becomes

$$2\underline{\mathbf{q}}^{(j)}\cdot\hat{\mathbf{r}}^{(j+1)} + \epsilon_{j+1}\,\hat{\not{R}}^{(j+1)}_{\epsilon_{j+1}}\,\vec{\gamma}\cdot\underline{\mathbf{q}}^{(j)}\,.$$

Eqs. (B.3) and (B.14) tell us that the first term on the R.H.S is of $O[g^2]$ in the region (B.6). The second term is of the form

$$\hat{\not{R}}^{(j)}_{\epsilon_j} \times O[g] + O[g^2] = \hat{\not{Q}}^{(j)}_{\epsilon_j} \times O[g] + O[g^2]\,.$$

Thus, we have

$$\text{Eq. (B.13)} = \hat{\not{Q}}^{(j)}_{\epsilon_j} \times O[g] + O[g^2]\,.$$

For $2 \leq j \leq n$, $\hat{\not{Q}}^{(j)}_{\epsilon_j} \times (\text{B.13}) = O[g^2]$.

The third entry is

$$
\begin{aligned}
&\mathcal{I}_Q^{(1)\,\sigma_j} \otimes \mathcal{I}_R^{(2)\,\rho_j}\; K_{j\rho_j}\,K_{j\sigma_j} \\
&\qquad \ni \left[\hat{\not{Q}}^{(j)}_{\epsilon_j} - \epsilon_{j+1}\vec{\gamma}\cdot\underline{\mathbf{q}}^{(j)}\right]\,\gamma^{\sigma_j}\,\hat{R}^{(j+1)\,\rho_j}_{\epsilon_{j+1}}\,K_{j\rho_j}\,K_{j\sigma_j} \\
&\qquad = \left[\hat{\not{Q}}^{(j)}_{\epsilon_j} - \epsilon_{j+1}\vec{\gamma}\cdot\underline{\mathbf{q}}^{(j)}\right]\,\hat{\not{K}}_j\,(\hat{R}^{(j+1)}_{\epsilon_{j+1}}\cdot K_j)\,.
\end{aligned}
\tag{B.15}
$$

In the region (B.6), $K_j = \epsilon(k_j^0)\, k_j\, \hat{K}_j + O[g^2T]$, where $\hat{K}_j = (1, \epsilon(k_j^0)\, \hat{\mathbf{k}}_j)$. Then, we obtain

$$
\begin{aligned}
\hat{R}^{(j+1)}_{\epsilon_{j+1}} \cdot K_j &= \epsilon(k_j^0)\, k_j\, \hat{R}^{(j+1)}_{\epsilon_{j+1}} \cdot \hat{K}_j + O[g^2T] \\
&= \epsilon(k_j^0)\, k_j \left[1 - \epsilon_{j+1}\, \epsilon(k_j^0)\, \hat{\mathbf{k}}_j \cdot \hat{\mathbf{r}}^{(j+1)}\right] \\
&\quad + O[g^2T] \\
&= O[g^2T]\,,
\end{aligned}
\tag{B.16}
$$

where use has been made of (B.14), (B.2), (B.3), and (B.6). The remainder of (B.15) may be analyzed as in the second entry above and we obtain

$$
\text{Eq. (B.15)} = \hat{\not{Q}}^{(j)}_{\epsilon_j} \times O[g^3T^2] + O[g^4T^2]\,.
$$

We recall that the denominator of the hard-gluon propagator that accompanies to (B.15) is $O[g^2T^2]$ smaller than those accompanying to (B.10) and (B.13) above (cf. (A.9) - (A.11)).

$\mathcal{I}_Q^{(2)\,\sigma_j} \otimes \mathcal{I}_R^{(1)\,\rho_j}$, Eq. (B.7), may be analyzed similarly as in the case of $\mathcal{I}_Q^{(1)\,\sigma_j} \otimes \mathcal{I}_R^{(2)\,\rho_j}$ and the same conclusion results.

3. $\mathcal{I}_Q^{(2)} \otimes \mathcal{I}_R^{(2)}$

Finally we analyze $\mathcal{I}_Q^{(2)\,\sigma_j} \otimes \mathcal{I}_R^{(2)\,\rho_j}$ in (B.7). The first entry is

$$
\begin{aligned}
&\mathcal{I}_Q^{(2)\,\sigma_j}\, \mathcal{I}_R^{(2)\,\rho_j}\, \mathcal{P}_{T\,\rho_j\sigma_j}(\hat{\mathbf{k}}_j) \\
&\quad \ni \hat{\mathbf{q}}^{(j+1)} \cdot \hat{\mathbf{r}}^{(j+1)} - (\hat{\mathbf{q}}^{(j+1)} \cdot \hat{\mathbf{k}}_j)(\hat{\mathbf{k}}_j \cdot \hat{\mathbf{r}}^{(j+1)}) \\
&\quad = O[g^2]\,.
\end{aligned}
$$

The second entry is

$$
\begin{aligned}
\mathcal{I}_Q^{(2)\,\sigma_j}\, \mathcal{I}_R^{(2)\,\rho_j}\, g_{\rho_j\sigma_j} &\ni \hat{Q}^{(j+1)}_{\epsilon_{j+1}} \cdot \hat{R}^{(j+1)}_{\epsilon_{j+1}} \\
&= 1 - \hat{\mathbf{q}}^{(j+1)} \cdot \hat{\mathbf{r}}^{(j+1)}\,,
\end{aligned}
$$

which, according to (B.14), is negligibly small.

The third entry is

$$\mathcal{I}_Q^{(2)\,\sigma_j}\,\mathcal{I}_R^{(2)\,\rho_j}\,K_{j\rho_j}\,K_{j\sigma_j} \ni (\hat{Q}^{(j+1)}_{\epsilon_{j+1}}\cdot K_j)(\hat{R}^{(j+1)}_{\epsilon_{j+1}}\cdot K_j)$$
$$= O[g^4T^2]\,,$$

where use has been made of (B.16).

This completes the proof of the statement made at the beginning of this Appendix.

APPENDIX C BRIEF ANALYSIS OF FIG. 9

Here we briefly analyze Fig. 9, where Q_1 $(= Q + K')$ and Q_2 $(= Q + K)$ are hard, all the three gluon lines carry soft momenta, K_1, K_2, and K_3 $(= -K_1 - K_2)$, and show that its contribution is nonleading.

The effective soft-gluon propagator, ${}^*\!\Delta^{\xi\zeta}_{ij}(K)$, consists of two terms, the one is proportional to $n_B(k_0) \simeq T/k_0$ $(= O(1/g))$and the other is independent of $n_B(k_0)$. The former term is of $O(1/(g^3T^2))$, while the latter term is of $O(1/(g^2T^2))$. As in the case of bare thermal propagator, the former term is independent of thermal indices, i and j.

For a given set of thermal indices $(i_1 - i_3, j_1 - j_3)$ in Fig. 9, we assign, on trial, the leading part of ${}^*\!\Delta^{\xi\zeta}_{ij}$ $(\sim O(1/(g^3T^2)))$ to the three gluon propagators. It can be shown explicitly that no divergence arises. In similar manner as in Sec. IV and Appendix B, we can estimate the order of magnitude of other characters in Fig. 9: ${}^\diamond\! S^{(\tau)}_{1i_1}(R + K_1)$, ${}^\diamond\! S^{(\tau)}_{i_31}(Q + K_1)$, and ${}^\diamond\! S^{(\tau)}_{i_1i_3}(Q - K_2)$ are of $O(1/\tilde{\gamma}_q)$. A tri-gluon vertex is of $O(gT)$. $\int d^4K_1 = \int dk_{10}\,dk_1\,k_1^2\,d(\hat{\mathbf{k}}_1\cdot\hat{\mathbf{q}}) = O\{(gT)^4\} \times O\{\tilde{\Gamma}/(gT)\} = O\{(gT)^3\tilde{\Gamma}\} = \int d^4K_2$. After all this, we see that, aside from a possible factor of $\ln(g^{-1})$, Fig. 9 is of $O(1)$, the same order of magnitude as the bare photon-quark vertex.

Now we note that, as mentioned above, the leading part of ${}^*\!\Delta^{\xi\zeta}_{ij}(K)$ is independent of the thermal indices and the three-gluon vertex with a blob in Fig. 9 may be written as

$$\mathcal{V}_{j_1j_2j_3} = g\left[(-)^{j_1}\delta_{j_1j_2}\delta_{j_1j_3}\mathcal{V}^{(0)} + \mathcal{V}^{(\mathrm{HTL})}_{j_1j_2j_3}\right]\,,$$

where the Lorentz indices are deleted. The first term comes from the bare vertex and the second term represents the HTL contribution. Recalling the identity

$\sum_{j_1,j_2,j_3=1}^{2} \mathcal{V}_{j_1 j_2 j_3}^{(\mathrm{HTL})} = 0$, we see that, upon summation over thermal indices, j_1, j_2 and j_3, in Fig. 9, the contribution under consideration vanishes.

This proves that the contribution of Fig. 9 to the production rate is nonleading.

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Figure captions

Fig. 1. Diagram that yields leading contribution to the soft-photon production rate in HTL-resummation scheme. "1" and "2" designate the type of photon-quark vertex. P, K, and K' are soft and the blobs on the solid lines indicate the effective quark propagators and the blobs on the vertices indicate the effective photon-quark vertices.

Fig. 2. HTL of the photon-quark vertex. "ℓ", "i", and "j" are thermal indices.

Fig. 3. Thermal self-energy part of a hard quark.

Fig. 4. "Modified" HTL of the photon-quark vertex. The square blobs on the solid lines indicate self-energy-part resummed hard-quark propagators, ${}^{\diamond}S$s.

Fig. 5. "Modified" HTL of the quark self-energy part. The square blob on the curly line indicates ${}^{\diamond}\Delta^{\mu\nu}(Q)$.

Fig. 6. An n-loop ladder diagram for the photon-quark vertex function. Solid lines stand for quarks and dashed lines stand for gluons.

Fig. 7. Nonladder diagram for the photon-quark vertex function.

Fig. 8. A "correction" to the HTL of the photon-quark vertex. The square blob on the vertex indicates $\hat{\Lambda}^{\mu}(Q+K', Q+K)$.

Fig. 9. A two-loop contribution to a photon-quark vertex.

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